Comparing and Scaling—Teacher Guide

Overview

Many quantitative problems can be solved by simply counting members of a set or by measuring segments, areas, volumes, angles, masses, or temperatures. However, it is often necessary to make decisions that involve comparisons of counts or measurements. The basic step in this kind of thinking is developed in elementary grades when such comparisons are decided by finding which number is largest. However, more useful reasoning often requires more careful comparison—explaining how much larger one number is than another, not in an absolute sense, but in a relative sense. There are many standard ways to make such comparisons—fractions, ratios, rates, differences, and percents. One of the fundamental goals of school mathematics, especially middle-grades mathematics, is helping students develop flexible understanding, skill, and disposition in using strategies for comparing quantities. This runs throughout the problems, ACE, and reflections of this unit. The unit confronts students with a series of mathematical tasks that encourage them to make decisions about the quantities relevant to each task, how those quantities can be compared most usefully, and what information is provided by various quantitative comparisons.

The second major theme of this unit, as the title suggests, is scaling. In its most familiar sense, scaling suggests making something bigger or smaller, but similar in key respects to an original. Ratios and fractions often express comparative information in scaled-down form. For example, if a class consists of 15 boys and 10 girls, we might say that the ratio of boys to girls is 3 to 2 or that $\frac{3}{5}$ of the class is boys. We could also say that 60% of the class is boys, a kind of scaling up.

*Stretching and Shrinking* lays a solid foundation of visual imagery to support the basic notion of scaling. The idea of ratio comparison was introduced there, along with informal ideas of equivalent ratios. These ideas are extended in the current unit. In addition rate is defined as a special ratio that compares two measurements with different units.

In *Stretching and Shrinking*, the problem was finding dimensions of a larger (or smaller) physical or graphical model of some situation while preserving the relative size of the component parts so that the figures remained mathematically similar. The same ideas and ways of thinking developed in *Stretching and Shrinking* become powerful ways of thinking about ratios. The goal is the same in many ratio situations—to scale the ratios up or down to determine whether they are the same or different.
Another way the more numerical situations in *Comparing and Scaling* relate to the ideas in *Stretching and Shrinking* is that comparison problems often call for finding the missing part of a ratio equivalent to a given ratio, which is the same as solving a proportion. For example, suppose you have a rectangle with dimensions of 5 cm by 7 cm. You want to draw a larger, similar rectangle with the dimension corresponding to 5 cm being 15 cm. What would the other dimension be? This is identical to the question posed above: If roses are 5 for $7 and I want to buy 15 roses, how much will they cost? In each case, we are dealing with the given ratio of 5 to 7 and looking for the equivalent ratio of 15 to $x$. *Stretching and Shrinking* precedes *Comparing and Scaling* to give students prior experience with these ideas in a more concrete geometric context.

To summarize, the broad purposes of this unit are to develop students’ ability to make intelligent comparisons of quantitative information—using ratios, fractions, decimals, rates, unit rates, and percents—and to use quantitative comparison information to make larger or smaller scale models of given situations or to scale rates and ratios up and down as needed. An additional goal of this unit is not only to have students learn different ways to reason in proportional situations, but to recognize when such reasoning is appropriate.

Many important mathematical applications involve comparing quantities of one kind or another. In some cases, the problem is simply deciding which of two quantities is the greater and describing how much greater it is. In such instances, we subtract to find a difference. This is what students deal with in elementary school. In fact, since situations that call for comparison by addition or subtraction come first in students’ experiences with mathematics, for many students this way of thinking becomes inappropriately pervasive in any situation requiring comparison.

**Rationale for Revision**

CMP2’s *Comparing and Scaling* has been altered in response to several forces, some internal and some external. The internal motivation for change was a desire to more strongly connect to, and to more completely develop, the ideas and strategies initiated in CMP3 *Stretching and Shrinking*. In similar figures, internal ratios, such as length: width, and external ratios, such as side length X: corresponding side length Y, are constant. When students find a missing length in *Stretching and Shrinking*, they are solving a proportion; they may use a scale factor rather than set up a proportion, but the proportional reasoning is the same. By the end of CMP3 *Stretching and Shrinking* students will have a method of scaling a ratio up or down to find a missing quantity, which is an important strategy for solving any proportion. Learning to choose the scale factor efficiently, in similarity and other situations, is a refinement of this strategy. In CMP3 *Comparing and Scaling*, students use proportional reasoning in contexts other than geometric
contexts, and develop additional strategies for solving proportions, including efficient scaling and common denominators. They will see that rate tables are a variation on a scaling strategy, and that unit rates are particularly useful. In the example below,

\[ \frac{1.5}{1} = \frac{x}{11} \]

number of units of length of base of triangle: 1 unit length of height of triangle = 1.5: 1. Using this unit rate is another way to solve the proportion 3:2 = x: 11.

The external motivations for changing *Comparing and Scaling* were caused by the requirement, in CCSSM, of the study of *ratio* in Grade 6, and the addition of *constant of proportionality* to the content in grade 7. Because students are expected to have some understanding of ratio when they enter grade 7, less time is needed to introduce ratios and rates. Therefore, more time can be spent on developing proportional reasoning and connections among rates, ratios, rate tables and proportions. In addition, since the *constant of proportionality* implies connections among tables, graphs and equations of proportional relationships, more can be done with these representations. This in turn provides connections to the next unit, *Moving Straight Ahead*, where students will see that equations of proportional relationships are one kind of linear equation, and the constant of proportionality is, in fact, the slope of the line, or the rate of increase in the table.
Comparing and Scaling Unit Goals

Goals of the Unit

Ratios, Rates and Percents: Understand ratios, rates and percents.

• Use ratios, rates, fractions, differences, and percents to write statements in a given situation comparing two quantities
• Distinguish between and use both part-part and part-whole ratios in comparisons
• Use percents to express ratios and proportions
• Recognize that rate is a special ratio that compares two measurements with different units.
• Analyze comparison statements made about quantitative data for correctness and quality
• Make judgments about which statements are most informative or best reflect a particular point of view (for example, a percent and a fraction comparison or a difference and a ratio)

Proportionality: Represent and Recognize Proportionality in Tables, Graphs and Equations

• Recognize that constant growth in a table or in a graph is related to proportional situations
• Write an equation to represent the pattern in a table of proportionally related variables
• Connect a unit rate and constant of proportionality to the equation describing a proportional situation
• Connect unit rate and constant of proportionality to a graph representing a proportional situation.

Reasoning Proportionally: Strategies for Solving Problems

• Recognize situations in which proportional reasoning is appropriate to solve the problem
• Scale a ratio, rate, percent, or fraction to make an equivalent ratio or rate (including unit rate), or to make a comparison.

• Strategically find equivalent rates (including unit rates) and ratios to solve problems

• Apply various strategies to solve for the unknown part when one part of two equal ratios is unknown, including scaling, rate tables, unit rates, equivalent ratios.

• Set up and solve proportions that arise in applications, for example, finding percents in the context of discounts and markups, converting measurement units.
Common Core State Standards (CCSS) Addressed in Comparing and Scaling

7.RP Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction 1/2/1/4 miles per hour, equivalently 2 miles per hour. \textit{Investigation 2, 3}

2. Recognize and represent proportional relationships between quantities. \textit{Investigation 2, 3}
   
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   
   c. Represent proportional relationships by equations. For example, if total cost \( t \) is proportional to the number \( n \) of items purchased at a constant price \( p \), the relationship between the total cost and the number of items can be expressed as \( t = pn \).
   
   d. Explain what a point \((x, y)\) on the graph of a proportional relationship means in terms of the situation, with special attention to the points \((0, 0)\) and \((1, r)\) where \( r \) is the unit rate.

3. Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error. \textit{Investigation 1, 2, 3}

7.EE. Use properties of operations to generate equivalent expressions.

2. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. For example, \( a + 0.05a = 1.05a \) means that “increase by 5%” is the same as “multiply by 1.05.” \textit{Investigation 3}
Table of Contents

Comparing and Scaling
Rate, Ratio, Proportion and Percent

Unit Opener

Mathematical Highlights

Unit Project: Conducting an Experiment

Investigation 1 Ways of Comparing: Ratios and Proportions
1.1 Surveying Opinions: Analyzing Comparison Statements
1.2 Mixing Juice: Comparing Ratios
1.3 Time to Concentrate: Scaling Ratios
1.4 Keeping Things in Proportion: Scaling to Solve Proportions

ACE Homework

Mathematical Reflections 1

Investigation 2 Comparing and Scaling Rates
2.1 Sharing Pizza: Comparison Strategies
2.2 Comparing Pizza Prices: Scaling Rates
2.3 Finding Costs: Unit Rate and Constant of Proportionality

ACE Homework

Mathematical Reflections 2

Investigation 3 Markups, Markdowns and Measures: Using Ratios, Percents, Proportions
3.1 Commissions, markups and discounts: Proportions with Percents
3.2 Measuring to the Unit: Measurement Conversions
3.3 Mixing it Up: Connecting Ratios, Rates and Percents
ACE Homework

Mathematical Reflections 3

Unit Project Paper Pool
Summary of Investigations

Investigation 1
Investigation 1 focuses on different strategies for comparing quantities—using ratios, fractions, percents. Students learn what different kinds of comparative statements mean about the data that is given. They are asked to write comparative statements that describe data. Questions are asked that engage students in making comparisons and checking the accuracy of statements given. The important question of how you decide whether to use a difference, ratio, fraction, or percent to make a particular comparison is raised. Building on the variety of strategies for making comparisons, students are asked to see how information in each of these forms provides the information needed to derive either of the other forms. Students investigate in more depth how ratios can be formed and scaled up or down to find equivalent ratios. This investigation more directly raises issues with comparison by finding differences.

Investigation 2
There are different ways to make comparisons and in this investigation students will reason about making choices about what to use in a given situation. The comparisons in this investigation motivate the creation and use of unit rates. Rate tables and unit rates provide other strategies for students to solve problems in proportional situations. Students learn to recognize proportional relations from a graph or table.

Investigation 3
In this investigation, students will be able to use the various proportional reasoning strategies they developed in two previous investigations into different contexts. Students will be able to practice conversions in different measurement units by scaling up or down one unit into another. They look for connections among the representations of proportional relationships, and among their solution strategies.
Launching the Unit
Introducing Your Students to Comparing and Scaling

One way to introduce your students to this unit is ask them to think about advertising slogans, like “2 out of 3 dentists advise their patients to chew sugarless gum.”

How do you think this information was collected?
Did the gum company really ask 3 dentists?
What do you think the gum company wants potential customers to know?
What would be another way to get the same point of view over to customers?

Having focused students on ratios and alternative ways to state the “2 dentists: 3 dentists” ratio, you might turn their attention to the three introductory questions in the SE.

Is there a ratio in the question about prices of pasta? What ratios are we comparing?
Is there a ratio in the question about selling used cars?
Is there a ratio in the question about chimp food?

How are these ratios like/not like the ratios you used in Stretching and Shrinking?

Close this discussion and transition to the first investigation by making the point that, in this unit, students will review and extend their knowledge about ratios and about scaling. You might point explicitly to the Mathematical Highlights page that follows the unit opening questions.

Mathematical Highlights

The Mathematical Highlights page in the student edition provides information to students, parents, and other family members. It gives students a preview of the mathematics and some of the overarching questions that they should ask themselves while studying Comparing and Scaling.

Using the Unit Project

The suggested unit project will ask students to investigate the pattern of bounces for a pool ball, as it makes its way around pool tables of various dimensions. As you might expect, ratio and scaling are involved in this investigation. The project is formally assigned near the end of the unit. We recommend that students work on the project with a partner. A scoring rubric and samples of student work are given in the Assessment Resources section.
Comparing and Scaling

Pacing/Management Chart

<table>
<thead>
<tr>
<th>Investigation &amp; Problems</th>
<th>Pacing</th>
<th>Materials</th>
<th>Vocabulary</th>
<th>Application</th>
<th>Connection</th>
<th>Extensions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Investigation 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>1 day</td>
<td>Display Sheet 1.1</td>
<td>ratio</td>
<td>1-9</td>
<td>32-38</td>
<td>68-69</td>
</tr>
<tr>
<td>1.2</td>
<td>1 day</td>
<td></td>
<td></td>
<td>10-12</td>
<td>39-43</td>
<td>70</td>
</tr>
<tr>
<td>1.3</td>
<td>1 day</td>
<td></td>
<td>Scaling</td>
<td>13-18</td>
<td>44-57</td>
<td>71-76</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>Part-to-part</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>Part-to-whole</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>1 day</td>
<td>Labsheet 1.4</td>
<td>Proportion</td>
<td>19-31</td>
<td>58-67</td>
<td>77-80</td>
</tr>
<tr>
<td>Mathemathical Reflection</td>
<td>½ day</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Check-up 1</td>
<td></td>
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<td></td>
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</tbody>
</table>

| **Investigation 2**      |        |           |            |             |            |            |
| 2.1                      | 1 day  | Display Sheet 2.1 | 1-3     | 14-17       | 27         |
| 2.2                      | 1 day  | Labsheet 2.2 | rate      | 4-8         | 18-23      | 28         |
| 2.3                      | 2 days | Display Sheet 2.3 | Unit rate | 9-13        | 24-26      | 29         |
|                          |        |           | Constant of proportional relationship |            |            |            |
|                          |        |           | Proportional relationship |            |            |            |
| Mathemathical Reflection | ½ day  |           |            |             |            |            |
| Partner Quiz             |        |           |            |             |            |            |

| **Investigation 3**      |        |           |            |             |            |            |
| 3.1                      | 1 day  | Labsheet 3.1 | Commission Markup | 1-8     | 17-20      | 36         |
| 3.2                      | 1 day  | Display Sheet 3.2 | 9-12   | 21-32       | 37         |
| 3.3                      | 1 day  | Display Sheet 3.3 | 13-16  | 33-35       | none       |
| Mathematica Reflection | ½ day       |               |               |               |               |
| Check up 2             |             |               |               |               |               |
| Unit project           | Optional    |               |               |               |               |
| Self assessment        | Take home   |               |               |               |               |
| Unit test              |             |               |               |               |               |
| Total                  |             |               |               |               |               |
Comparing and Scaling—Mathematics Background

The subtitle of Comparing and Scaling is Ratios, Rates, Percents, Proportions. This subtitle makes clear that the heart of the unit goals is to recognize when making comparisons using these strategies is appropriate, then to use these strategies with understanding and efficiency.

Scaling Ratios as a Strategy

To compare two or more related measures or counts, such as 3 roses for $5 and 7 roses for $9, you need strategies that allow the related pairs of numbers to be compared. Simple subtraction will not tell you what you want to know. Understanding of ratio and proportion is needed. A proportion is a statement of equality between two ratios. In this example, you need to find a way to scale the ratios of 3 to 5 and 7 to 9 so that they can be directly compared. Many students think these two ratios are the same, reasoning that 4 has been added to each of the numbers 3 and 5 to get 7 and 9. This is an example of students’ misconceptions about when additive comparisons are appropriate. If you appropriately scale both ratios so that either the number of roses or the costs are the same, you can make a simple comparison of the quantities that are not the same. The two possibilities are shown below:

If you want to scale the costs to be the same, the kind of thinking is the same as that for finding a common denominator: look for a number that represents a multiple of the two numbers 5 and 9. If you scale to make the prices the same (that is, $45), then the answer is immediately obvious.

\[
\frac{3}{5} = \frac{3 \times 9}{5 \times 9} = \frac{27}{45} \quad \text{and} \quad \frac{7}{9} = \frac{7 \times 5}{9 \times 5} = \frac{35}{45}
\]

You can now compare the ratios 27 roses for $45 and 35 roses for $45. Clearly the second option gives more for the same amount of money.

Let’s scale the numerators to be the same.

\[
\frac{3}{5} = \frac{3 \times 7}{5 \times 7} = \frac{21}{35} \quad \text{and} \quad \frac{7}{9} = \frac{7 \times 3}{9 \times 3} = \frac{21}{27}
\]

You can now compare the ratios 21 roses for $35 and 21 roses for $27. Again the best buy is obvious.

This example underscores the relationship between the mathematical thinking used to find common denominators or common numerators in work with equivalent fractions and that was used to find equivalent ratios. Ratios are written in several forms. Some of the most often used are 2 to 3, or 2 : 3, or \(\frac{2}{3}\). In the example, the convenience of writing the ratios as fractions supports the thinking needed for scaling up the ratios. However, we must make sure that students can differentiate between a ratio written as a fraction and a fraction representing the fractional part of a whole. We address this in the next section.

Using Ratio Statements to Find Fraction Statements of Comparison

The statement “the ratio of girls to boys in a class is 15 girls to 9 boys” can be written as the fraction \(\frac{15}{9}\), but it does not mean that the fraction of students in the class that are girls is \(\frac{15}{9}\). This is confusing for students and leads some teachers to avoid the fraction form for writing a ratio. We have chosen to confront the confusion by asking the fraction question directly.
Maria says the fraction of the class that is girls is \( \frac{15}{9} \). Bob says the fraction of the class that is girls is \( \frac{15}{24} \). Who is correct and why?

The correct answer hinges on recognizing that a new quantity is actually used to find the fraction of students in the class that are girls. The total number of students in the class is needed. This is the sum of the numbers of boys and girls, 24. The part to whole comparison is \( \frac{15}{24} \), and Bob is correct. Now we turn to another strategy for solving the roses problem.

**Per Quantities: Finding and Using Rates and Unit Rates**

If you compute the price per rose, you will have a rate comparison for the roses problem. In the 3 for $5 deal, the unit rate is $1.67. The price per rose in the 7 for $9 deal is $1.29—clearly the better price. Alternatively, at the 3 for $5 price, 7 roses would cost $11.67. This is a different comparison with the same result. Let’s explore this strategy a bit further.

Here are two ratios that suggest rates:

* Sascha goes 5 miles in 20 minutes on the first part of his bike ride. On the second part, he goes 8 miles in 24 minutes. On which part is he riding faster?

Many students will intuitively want to divide the miles number and the minutes number to get a result, but they sometimes lose track of which one is divided into the other. Consequently they produce a number, but have no idea what the number means in the problem. Here the comparison can be made in two different ways by computing different unit rates. Let’s look at each.

Suppose a student decided to divide 5 by 20 and 8 by 24. She gets the two numbers 0.25 and 0.333. What do these numbers mean? She might have divided 20 by 5 and 24 by 8. This division gives 4 and 3. What do these numbers mean? You have to know before you can decide what they tell us about the two legs of the bike ride. So start again and this time carry the label with the quantities.

\[
\frac{5 \text{ miles}}{20 \text{ minutes}} = 0.25 \text{ miles per minute} \quad \text{and} \quad \frac{8 \text{ miles}}{24 \text{ minutes}} = 0.333 \text{ miles per minute}
\]

Now the comparison is clear. The times are the same, 1 minute, and the distances can be directly compared. 8 miles in 24 minutes is faster.

But, you could divide the other way:

\[
\frac{20 \text{ minutes}}{5 \text{ miles}} = 4 \text{ minutes per mile} \quad \text{and} \quad \frac{24 \text{ minutes}}{8 \text{ miles}} = 3 \text{ minutes per mile}
\]

Now you see that the lesser number tells the correct answer, 8 miles in 24 minutes.

**Rate tables** are very helpful in keeping clear what these divisions mean. The units are visible, and scaling “to 1” is done in a way that makes the division clear.

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<td>T minutes</td>
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Scaling 20 minutes to make 1 minute, we divide by 20. So scaling 5 miles in the same way we get 0.25 miles. Likewise, scaling 5 miles to make 1 mile we divide by 5. So scaling 20 minutes the same way we get 4 minutes.

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<th>0.25</th>
<th>1</th>
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<tr>
<td>T minutes</td>
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What makes unit rates so interesting, and somewhat difficult, for students is that you have two options when you divide two numbers. The units help students think through such situations with the goal of building the flexibility to use either set of unit rates to compare the quantities.

One of the recurring themes of these materials is that we can represent data in different ways and that each way may tell us something that is not as obvious from other representations. The comparison in the rose example can be made in several ways: for example, using unit rates, comparing the ratios in fraction form to determine which is greater, or scaling both rates until the price is the same or the number of roses is the same. Developing strategies for deciding what the comparison situation calls for and for making comparisons are major goals of this unit.

Relating Ratios, Fractions and Percents

It is often desirable to change one form of comparison statement to another. The question is, can you write a percent statement given either a ratio or a fraction statement, and can you write a ratio or fraction statement given a percent comparison statement? Let’s explore this with an example.

The ratio of concentrate to water in a mix for lemonade is 3 cups concentrate to 16 cups water. The questions you might ask are: “What fraction of the lemonade will be concentrate?” or “What percent of the lemonade will be concentrate?” First find the total cups the recipe makes. It makes 19 cups. Then write the fraction of the lemonade that is concentrate, $\frac{3}{19}$. Now finding the percent is easy. Just divide the concentrate by the total, $3 \div 19 = 0.15789…$ or about 15.8% concentrate.

Suppose you know that the percent of boys in a class is 48% and you want to write this as a ratio. You can think of the percent as a scaling of the ratio representing boys and girls up to a total of 100. So the girls are 52% of the class and the ratio of boys to girls is 48 to 52. You can scale this ratio down to 12 boys to 13 girls. The powerful thing about these related representations is the flexibility it gives us to choose the form of representation that describes the situation best for our purposes.

One caution about such changes of representation is that the choice to make these changes of form should be judged against whether the computations you do will have meaning. For example, in many rate situations, such as miles per gallon, trying to compute a percent does not make sense because the addition to get a total does not make sense. Miles covered plus gallons of gas used is a meaningless total. When the ratio can be thought of as part of a whole, the change of form we described makes sense (for example, white paint to blue paint in a mix, or high-fiber to high-protein nuggets in food for a baby chimp).

Proportions and Proportional Reasoning

The related concepts and skills in this unit are often referred to as proportional reasoning. Forming ratios in order to make comparisons is the heart of proportional reasoning. What is a proportion? A proportion is simply a statement of equality between two ratios. What makes this idea powerful is that if we know a ratio is equivalent to another, but we do not know both terms of one of the ratios, we can use what we already know about scaling or
finding equivalent fractions to find the missing part of a proportion. Again, let’s look at an example.

_It takes Glenda 70 steps on the elliptical machine to go 0.1 mile. When her workout is done, she has gone 3 miles. How many steps has she taken on the machine?_

Here is a proportion and a solution for the number of steps that Glenda made.

\[
70 \text{ steps} \quad 0.1 \text{ miles} = x \text{ steps} \quad 3 \text{ miles}
\]

The first ratio in the proportion is scaled up by multiplying both the numerator and the denominator by 30. Thus, the denominator equals the denominator of the ratio with the unknown, \(x\). This allows us to read the value of \(x\) directly since we know that if the two fractions are equivalent and have the same denominator, the numerators are also the same. The strategy we use to find the number by which we multiply, or “scale,” is the same as the thinking process we use to find common denominators for fractions.

How far you go in formalizing the solving of proportions will depend on you and your students. We highly recommend that you do not impose solution strategies that have no meaning for the students. While cross multiplication is efficient, for most students at this level it is used without any understanding of why it works and consequently is often misused. We believe that students are better served by having the time to learn to think through situations requiring solving proportions and develop flexibility in approaching a problem so that easy possible solution strategies are not missed in a rush to an algorithm. This approach also builds on mathematics students already know and ways of thinking that they have already acquired. Helping students want to make sense of mathematics is encouraging a kind of thinking and flexibility that will allow them to feel confident to tackle problems that do not look exactly like ones they have already solved. Part of the goal of this unit is for students to learn to make judgments about the situation and to choose methods for comparing and for scaling.

**Student Scaling Strategies**

Building on strategies from _Stretching and Shrinking_ students will likely form an efficient strategy for scaling one of the ratios in a proportion to make an equivalent ratio, making it clear what the unknown part of the other ratio is. In 1.4D \(\frac{7}{12} = \frac{x}{9}\), students have to consider how they will scale the 12 to make 9, to facilitate a comparison. The scale factor that will do this is \(\frac{9}{12}\) or 0.75, because \(12 \times (9/12) = 9\). To complete scaling the ratio 7:12 to make an equivalent ratio they now have to multiply 7 by the same scale factor,

\[
(7\times 0.75)/(12 \times 0.75) = x/9,
\]

so 5.25/9 = x/9, so x = 5.25.

The beauty of this efficient scaling strategy is that it never obscures that students are making equivalent ratios.

A variation on this efficient scaling strategy is to scale both ratios in a proportion to have a common denominator.

Once students have encountered rate tables they have another tool for finding a solution for a proportion, scaling to a unit rate first.

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<thead>
<tr>
<th>miles</th>
<th>7</th>
<th>7/12</th>
<th>x</th>
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</thead>
<tbody>
<tr>
<td>minutes</td>
<td>12</td>
<td>1</td>
<td>9</td>
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Cross Multiply

If cross-multiplication is mentioned, and if the students seem interested, don’t just give a procedure for cross multiplication. Develop the idea based on what your students already know—finding common denominators. (If we use the product of the original denominators as a common denominator, the numerators will be cross products). In Question D of Problem 1.4, \( \frac{7}{12} = \frac{x}{9} \), a common denominator could be 108. In the first fraction we have to multiply the numerator and denominator by 9. In the second fraction we need to multiply the numerator and the denominator by 12 to make the denominators the same.

Because the denominators are now the same, we need to find the value of \( x \) that makes the numerators equal. So we have to find \( x \) when 12\( x \) = 63. This means that \( x \) must be 5.25.

\( \frac{7}{12} = \frac{x}{9} \) is equivalent to \( \frac{63}{108} = \frac{12x}{108} \) is equivalent to 63 = 12\( x \), therefore \( x = 5.25 \).

So in a sense, cross-multiplying asks the question: What would be the numerator if these two fractions had a particular common denominator (the product of the original denominators)?

(Note: students could equally correctly choose 36 or 72 or 144 etc. as a common denominator in this example. Choosing to multiply the two original denominators makes the meaning of cross-multiplying clear, but there is nothing more correct or advantageous in choosing 108. Scaling to make a common denominator is a fine strategy, even if students never connect it to “cross multiplying.” In fact students may scale both ratios in a proportion to have a common numerator, instead of a common denominator, if that is more convenient.)

Helping students to make their own reasoning explicit can lead to a generalized method of solving proportions. For example, when many students solve this proportion \( \frac{3}{7} = \frac{x}{343} \), they do the following arithmetic.

\[
(343 \div 7) \times 3 = \frac{343}{7} \times 3 = \frac{343 \times 3}{7}
\]

The division 343 \( \div \) 7 finds the scaling factor by which we need to scale the 3.

Consequently, for solving a general proportion \( \frac{a}{b} = \frac{x}{c} \), we can follow the same reasoning: find the scaling factor by computing \( c \div b \), then multiply the scaling factor by \( a \). So the arithmetic we actually perform is scale factor \( \times \) the known numerator. In symbols this is \( \frac{c}{b} \times a = x \) or \( \frac{cx}{b} = x \).

With the unknown in the denominator, find the scale factor using the numerators so that we can scale the denominators to find the unknown. To solve \( \frac{a}{b} = \frac{x}{c} \), we first find the scale factor by which we can make the numerators the same, \( c \div a \). Then we have to scale the denominator to see what is equal to \( x \). This gives \( \frac{c}{a} \times b = x \) or \( \frac{cx}{b} = x \).

An alternative strategy can be built using fact family ideas. To solve \( \frac{a}{b} = \frac{x}{c} \), think of the equation as \( \frac{a}{b} = c \div x \). From fact families, you can say that \( x = c \div \frac{a}{b} \). Rewriting the right side with common denominators gives \( x = \frac{cb}{b} \div \frac{a}{b} = cb \div a \), or \( x = \frac{cb}{a} \).

These are the equations you would get by cross-multiplication, but here the explanation is built on students’ ways of reasoning.
**Constant of Proportionality and Graphs**

When two variables, X and Y, are proportionally related to each other, the relationship can be represented as the proportion \( Y/X = A/B \), where A and B are two related values of Y and X. This can be rewritten as \( Y = (A/B)X \) or \( Y = mX \), where \( m = A/B \) is the unit rate. (For example, if 10 oranges cost $1.90 how much will 13 oranges cost? The variables are \( X = \) number of oranges and \( Y = \) cost of X oranges. So, in this case \( A = 1.90 \) and \( B = 10 \). The equations would be \( Y = (1.90/10)13 \), where \( 1.90/10 \) gives the unit rate per orange.) This is a linear relationship, which students will learn more about in Moving Straight Ahead.

We have seen above that we can use a proportion or unit rate to predict values for X or Y, if we know values for a related pair of values, A and B. We can also use the equation \( Y = mX \) to predict values for X or Y. For example, in 2.2 students use a rate table to find the cost of 1 pizza at Lion’s Den, where 10 pizzas cost $120. They can predict the cost of any number of pizzas, say \( N = 9 \) pizzas, by extending the rate table, or by setting up a proportion, \( $120 : 10 \text{ pizzas} = \$D : 9 \text{ pizzas} \), or by using the equation \( D = 12N \). The coefficient of \( N \) in this equation is the unit rate per pizza; this is also called the constant of proportionality.

The graph of this equation represents all the pairs of values which fit the equation \( D = 12N \). Points on the graph represent all the solutions for the proportion \( $120 : 10 \text{ pizzas} = \$D : N \text{ pizzas} \); if you know the value of either \( N \) or \( D \) you can find the corresponding solution on this graph.
Notice that the graph passes through \((0, 0)\), which makes sense for Dollars and Pizzas. In fact for any relationship represented by an equation of the form \(Y = mX\) the graph will be a straight line passing through \((0, 0)\); this is one of the ways students can recognize a proportional relationship from its graph.

The table representing a proportional relationship also has distinguishing characteristics. In 2.2 students compare the tables for the costs of pizzas with a fixed delivery cost and without. The patterns they see in the “without delivery” costs, like being able to double the cost of 5 pizzas to find the cost of 10 pizzas, or adding the same amount to the cost for every 1 pizza added, echo the underlying scalability of the relationship. This scalability is not present for the “with delivery” costs; at $13 each and a delivery charge of $5, 5 pizzas cost \(5 \times 13 + 5\), or $70, and 10 pizzas cost \(10 \times 13 + 5\), or $135. (The graph of \(D = 13N + 5\) does not pass through \((0,0)\).

**Connections to Other Units**

<table>
<thead>
<tr>
<th>Big Idea</th>
<th>Prior Work</th>
<th>Future Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understand ratios, rates and percents.</td>
<td>exploring and applying rational number concepts (<em>Bits and Pieces I, II and III</em>)</td>
<td>calculating and applying slope in equations of (y = mx + b) form (<em>Moving Straight Ahead and Say It With Symbols</em>)</td>
</tr>
<tr>
<td>Understand ratios, rates and percents.</td>
<td>percent defined as a ratio to 100 and connected to fractions and decimals (Bits and Pieces I, II and III)</td>
<td>making comparisons between groups of different sizes (Data Around Us)</td>
</tr>
<tr>
<td>Understand ratios, rates and percents.</td>
<td>fractions as a part/whole comparison, addition, subtraction, multiplication, and division with fractions (Bits and Pieces I &amp; II, What do you Expect?)</td>
<td>expressing and applying probabilities as fractions (What Do You Expect?), determining if two algebraic expressions are equivalent (Say It With Symbols)</td>
</tr>
<tr>
<td>Represent and Recognize Proportionality in Tables, Graphs and Equations</td>
<td>comparing and subdividing similar figures to determine scale factors (Stretching and Shrinking)</td>
<td>scaling up rectangular prisms (Filling and Wrapping)</td>
</tr>
<tr>
<td>Represent and Recognize Proportionality in Tables, Graphs and Equations</td>
<td>connecting and comparing rates using ratios, decimals, and percents (Bits and Pieces I &amp; II), comparing data sets (Data About Us)</td>
<td>comparing probabilities (What Do You Expect?, Data Around Us) Comparing data sets (Data Around Us)</td>
</tr>
<tr>
<td>developing strategies and techniques to solve for missing values in a proportion</td>
<td>making inferences about quantities and populations based on experimental or theoretical probabilities (What do You Expect?)</td>
<td>estimating with and comparing large numbers (Data Around Us), developing benchmarks and skills for estimating irrational numbers (Looking for Pythagoras), estimating populations (Data Around Us)</td>
</tr>
</tbody>
</table>
Investigation 1 Ways of Comparing

Mathematical and Problem-Solving Goals

Ratios, Rates and Percents: Understand ratios, rates and percents.

- Use ratios, rates, fractions, differences, and percents to write statements in a given situation comparing two quantities
- Distinguish between and use both part-part and part-whole ratios in comparisons
- Use percents to express ratios and proportions
- Recognize that rate is a special ratio that compares two measurements with different units.
- Analyze comparison statements made about quantitative data for correctness and quality
- Make judgments about which statements are most informative or best reflect a particular point of view (for example, a percent and a fraction comparison or a difference and a ratio)

Proportionality: Represent and Recognize Proportionality in Tables, Graphs and Equations

- Recognize that constant growth in a table or in a graph is related to proportional situations
- Write an equation to represent the pattern in a table of proportionally related variables
- Connect a unit rate and constant of proportionality to the equation describing a proportional situation
- Connect unit rate and constant of proportionality to a graph representing a proportional situation.

Reasoning Proportionally: Strategies for Solving Problems

- Recognize situations in which proportional reasoning is appropriate to solve the problem
- Scale a ratio, rate, percent, or fraction to make an equivalent ratio or rate (including unit rate), or to make a comparison.
- Strategically find equivalent rates (including unit rates) and ratios to solve problems
- Apply various strategies to solve for the unknown part when one part of two equal ratios is unknown, including scaling, rate tables, unit rates, equivalent ratios.
- Set up and solve proportions that arise in applications, for example, finding percents in the context of discounts and markups, converting measurement units.

Investigation 1
Mathematical Goals and Mathematical Reflections

<table>
<thead>
<tr>
<th>Goals</th>
<th>Mathematical Reflections</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make judgments about which statements are most informative or best reflect a particular point of view (for example, a percent and a fraction comparison or a difference and a ratio).</td>
<td>1. a. In this investigation you have used ratios, percents, fractions and differences to make comparison statements. How have you found these ideas helpful?</td>
</tr>
<tr>
<td>Distinguish between and use both part-part and part-whole ratios in comparisons.</td>
<td>1. b. Give examples to explain how part-to-part ratios are different from, but related to, part-to-whole ratios.</td>
</tr>
<tr>
<td>Recognize situations in which proportional reasoning is appropriate to solve the problem.</td>
<td>2. How do you use scaling or equivalent ratios?</td>
</tr>
<tr>
<td>Scale a ratio, rate, percent, or fraction to make an equivalent ratio or rate (including unit rate), or to make a comparison.</td>
<td>a. To solve a proportion? Give an example.</td>
</tr>
<tr>
<td>Strategically find equivalent rates (including unit rates) and ratios to solve problems.</td>
<td>b. To make a decision? Give an example.</td>
</tr>
<tr>
<td>Set up and solve proportions that arise in applications, for example, finding percents in the context of discounts and markups, converting measurement units.</td>
<td></td>
</tr>
</tbody>
</table>

Teaching Notes for Each Problem

1.1 Surveying Opinions: Analyzing Comparison Statements

Introduction

Description of Problem 1.1

The language of ratios is introduced as simply a way of phrasing comparisons. The setting is typical of preference data from advertisements. Students are likely to be familiar with advertisements such as these. Be sure to keep the conversation at the exploration level. Different kinds of ratio comparisons and ways of representing the comparison (fractions, ratios, rates, and percents) surface in the problem. Each of these will be developed more fully in later problems.

Focus Question:

What do different comparisons of quantities tell you about their relationship?

Grouping: Small groups of 3 to 4.

Investigation 1
Launch

Connecting to Prior knowledge
You may start this problem by asking the students about their experiences with advertisements. Some questions you could ask include:

* Can you think of any commercials that compare one product to another or several others?
* Can you think of any that involve a comparison using numbers?

You may want to bring in a product ad that involves such a comparison of numbers. Take stock of the comparison types with which students are most familiar. Turn to the four ways of presenting data from the cola trials. Read them with your students.

Suggested Questions
Challenge them to discuss in groups questions such as:

* What do you know from each form of comparison given?
* What information is missing from each form of comparison?
* Is each form of comparison accurate and effective?

Issuing the challenge
View these discussions as setting the scene for the exploration and the summary. Do not press too hard for the students to come up with good answers. The intent is to plant these kinds of questions in their heads as they begin their work on the problem. Students may see that, in the 3-to-2 comparison, you lose any sense of how many people were in the trials, but you can immediately get a sense of the comparison. Out of every five people in the trials, three liked Bolda Cola better. This latter point is one to push in the summary as some students will have difficulty realizing that 3 to 2 means 3 out of 5 and 2 out of 5.

Explore

Suggested Questions
This problem is meant to raise issues that will be studied in more detail in the unit. Even so be on the lookout for students who are having trouble moving beyond finding differences as the method of comparison. You will need to pay particular attention to these students as the unit progresses. If you notice this occurring you might ask students,

* What other methods of comparisons are there besides finding differences? How might those other methods be useful?

As you move from group to group, look for good explanations or insights about what each form of comparison does and does not provide. This activity should set the stage for the kinds of questions students should ask themselves as they solve problems by making comparisons.

One thing to press students on is how they can take one comparison and re-write it in another form. For example in part B, some questions you might ask include:

* If one third of the students prefer a concert to an athletic event, what fraction prefers an athletic event to a concert?
• Out of the 150 surveyed, how many preferred a concert and how many preferred an athletic event? (50 preferred a concert, and 100 preferred an athletic event.)

• Now write a ratio statement comparing those who prefer a concert to those who prefer an athletic event. (50 to 100)

• What is another ratio that is equivalent to this one that is easier to use? (1 to 2)

• How did you find this ratio? (Divided both 50 and 100 by the common factor of 50.)

• How is this like finding an equivalent fraction? (is equivalent to because we can divide both the numerator and the denominator by the common factor of 50 and keep the fraction equivalent.)

Going Further
• From the statement, “People prefer Bolda Cola over Cola Nola by a ratio of 3 to 2,” can you tell what fraction of those surveyed preferred Bolda Cola? (3/5, or 3 out of 5 prefer Bolda Cola.)

If students finish early, ask them to write an accurate comparison that is different from those given in parts A and B.

Summarize
As a class, discuss what the groups found. The common language used in making comparisons should, in this problem, be examined through the lens of mathematics. For each form of comparison, display for the class (e.g. write on the overhead) what it tells and what it does not tell about the newspaper ad. Be sure students see that some of the statements lose the original data and some preserve the data. However, some forms are easy to visualize and others are not. Again, this problem is for raising issues and reviewing everyday ways of making comparisons.

Suggested Questions
Here are some questions that will be useful in the Summary:

• From the first statement, can you tell how many people took the taste test? (Yes, 28,565 people took the taste test.)

• If you had only this statement, would you have a good sense of the strength of the preference for Bolda Cola? If not, what would you do with the statement to make it tell a better story about the test? (Many will say no and suggest finding the difference. If so, then go directly to Statement 2.)

• How is Statement 2 related to Statement 1? (It is the difference between those that preferred Bolda Cola and those that preferred Cola Nola.)

• If you did not have Statement 1, just Statement 2, could you tell how many people took the survey? (No.)

• Does this statement give you a sense of the strength of preference? (This depends on the number taking the survey. If 15,000 took the survey, this is a real difference. If 500,000 took the survey, this is not much of a difference.)

Investigation 1
• Does knowing that 60% preferred Bolda Cola give a good sense of the comparison? (Yes. This percentage tells you the comparison regardless of the number taking the survey. However, you do have to worry about whether the sample is representative of the entire population.)

• Do you know from this statement how many were surveyed? (No.)

• Do you know from the 3 to 2 statement how many were surveyed? (No.)

• How are the 60% and the 3 to 2 statement related? (60% means that 40% preferred the other cola. 60 to 40 is the same thing as 3 to 2.)

Questions to push students’ thinking

Check for Understanding (optional)

To push students to think a bit more about comparison statements and to check their understanding, use the situations and questions that are posed on Display Sheet 1.1B.

A school counselor considered several different ways to report on the seventh-grade class: The seventh-grade class in Neilson Middle School has more girls than boys.

• “The ratio of girls to boys is 5 to 4.”
• “Girls comprise about 56% of the class.”
• “Four ninths of the class members are boys.”

1. What does the word “ratio” mean? (A ratio of 5 to 4 means that for every 5 girls in the class, there are 4 boys.)

2. What does “56% of the class” mean? (56% means that if there were 100 students in the class, 56 of them would be girls.)

3. What does “four ninths of the class” mean? (Four ninths means that out of every 9 students in the class, 4 are boys.)

4. Could the counselor’s statements all be correct? Why or why not? (Yes, they are all equivalent statements. A ratio of 5 girls to 4 boys means that 4 out of 9 students are boys and 5 out of 9 students are girls. Five ninths is equivalent to 55.5…%, or about 56%.)

5. Can you tell how many students were in the class? Why or why not? (No, none of the statements tells you the exact number of girls, boys, or total students.)

Focus Question

What do different comparisons of quantities tell you about their relationship?

Answers to Problem 1.1

A. 1. Statement A means that out of the total number surveyed (17,139 + 11,426 = 28,565), 17,139 of them preferred Bolda Cola and 11,426 preferred Cola Nola. In other words, for every 17,139 people who preferred Bolda Cola, there were 11,426 who preferred Cola Nola. Statement B reports the difference in numbers between the two groups of people. We can’t tell how many people were surveyed, but we do know that 5,713 more people preferred Bolda Cola. Statement C means that 60 percent (60%) of the sample chose Bolda Cola as their preference. 60% means that if there were 100 people, 60 would prefer Bolda Cola. Statement D means that for every 3 people who preferred Bolda Cola, there were 2 people who preferred Cola Nola.

Investigation 1
2. The ratio 3 to 2 or the 60% might be the most effective advertisements because the numbers are smaller and easier to relate to. You can easily use the ratio of 3 to 2 to predict what you would expect preferences to be in your class or in some other group of people. Or, the greater numbers may make a more powerful impression; the difference between 3 and 2 is only 1, while the difference between 17,139 and 11,426 is 5,713.

3. Yes, it is possible. The ratio of 17,139 to 11,426 (which have a difference of 5,713) can be approximated as 3 to 2, which is equivalent to 60% of the people surveyed choosing Bolda Cola over Cola Nola. Notice that 60% = \(\frac{3}{5}\), not \(\frac{3}{2}\). It is important to keep asking whether we are comparing part-to-part or part-to-whole in a given situation.

4. 1.5 times as many people preferred Bolda Cola to Cola Nola. \(\frac{3}{5}\) (three fifths) of people surveyed preferred Bolda Cola.

5. 600 would prefer Bolda Cola, and 400 would prefer Cola Nola. 60% of 1,000 is 600, and 1,000 - 600 = 400.

B. 1. Yes. The total number of students is 150, and \(\frac{50}{100} = \frac{1}{3}\).

2. Yes. 100: 50 = 2 : 1

3. Yes. 50: 100 = 1 : 2

4. Yes. 100 - 50 = 50. The problem discusses difference.

5. Yes. 100 = 2 \times 50

6. No. About 33% of students prefer radio to television.

C. 1. Since all the correct statements involve ratios or fractions, they tell us how to compare the preferences of parts of the student population. Only 4 tells us how many students are in each of these parts; the introduction tells us how many students were polled.

2. You can scale up a part-to-whole ratio from a small sample, say 2 out of 5 people, so that the “whole” is the entire population, say 20,000 out of 50,000.

1.2 Mixing Juice: Comparing Ratios

Introduction

Description of Problem 1.2

Students encounter an open-ended situation in which comparison is needed to make a decision. In this problem, students are challenged to compare drink recipes. They use proportional reasoning to figure out how to make different mixes of orange juice for 240 people. The advantage to posing the problem in an open way will allow students to find many different ways to think about the situation. A discussion about ratios after the students have their own ideas about the problem is effective. Revisiting the problem with an eye on ratios can be done in the summary.

Focus Question:

What strategies do you use to determine which mix is the most orangey?

Grouping:

Have students work together in groups of three to four students.

Investigation 1
Launch

Connecting to Prior knowledge

In problem 1.1, students analyzed different comparison statements. These statements compared different quantities (e.g., part-to-whole, part-to-part) as well as in different forms (e.g., ratios, percents, exact or approximate values). In problem 1.2, students extend their thinking on comparing quantities in different ways to consider which orange juice recipe will be the most or least concentrated. Students should use statements about ratios, percents, and fraction statements to give information about situations. This helps to focus the students' attention on ratio comparisons and the ways that such comparisons are represented.

Suggested Questions

Launch Problem 1.2 by making sure that students understand the context.

- How many students have made orange juice from a can before?
- What was involved in making it?

Issuing the challenge

You may want to bring in a can of frozen orange juice (thawed) and, with your class, actually make juice, following the instructions on the can. You can discuss the fact that you have one container of concentrated juice and to this you add three containers of water (or whatever it says on the container of concentrate). Point out that the recipes given in the problem are different from the one on the can. At camp, the frozen concentrate comes in a very large container without mixing proportions given.

This is a challenging problem for students. Some teachers do a launch and then pull the class back together to discuss their initial thinking about the different recipes. This gives a chance for groups to hear different strategies. Together, Questions A and B of the problem can be discussed (focusing on the most and least “orangey” recipes), and then the groups can be challenged to solve Questions C and D of the problem.

Explore

As students are working on Questions A and B, circulate asking them questions. If a group has a conjecture about most and least orangey, ask them to explain why. Some students will use naïve strategies such as simply focusing on the number of cans of concentrate and ignoring the water.

Suggested Questions

Ask questions to challenge students’ ideas.

- Can I keep adding cans of water without making the juice less orangey? Other students will propose that \( \frac{2}{3} \) of the juice in Mix A is concentrate. Ask:

- How much juice is made from one recipe?

Leave the students to sort out what the actual fraction of the juice in each recipe is concentrate. Usually the students will begin to question their initial thoughts and recognize that the whole in Mix A is 5 cups so the fraction that is concentrate is \( \frac{2}{5} \).

Question C is designed to raise the issue of writing comparison statements using fractions from data such as this. The questions suggested above are appropriate here as well.

Investigation 1
For Question D, if students are stuck on the question of making a recipe to feed 240, ask them to consider Mix A. Have them review how much total liquid (concentrate and water) is in this batch (5 cups).

Use these questions as guides.

• What are your ideas about how many people one batch of this juice might serve?

• What if each serving is an 8 oz. glass, the same as 1 cup? (5 servings are possible)

• What if each serving is ½ cup? (10 servings are possible)

• If you were going to serve juice to 50 people, how many batches would you have to make if each person gets ½ cup of juice? (5 batches)

• What are different strategies you might use to answer this question? (Students might just divide 50 people by the 10 servings per batch to determine that 5 batches are needed. Alternatively, some students may reason that if 1 batch makes 10 servings, then 2 batches make 20 servings, 3 batches make 30 servings, 4 batches make 40 servings, and 5 batches make 50 servings. Students may choose to make a table from which to reason.)

You also may want to do some drawings to match this table. Students may want to sketch ten glasses to represent one batch. Then they can repeat these ten glasses for each batch, showing that the number of glasses is a multiple of 10 as the number of batches increases.

Question E asks for another kind of reasoning. Here, rather than making several batches, the students are asked to figure out how to make exactly one cup of juice that fits each of the recipes. In essence they have to partition accurately one whole into two parts—one part that represents the water and the other part that represents the concentrate. If a group is stuck, ask how many parts they will have to partition a cup into to make the cup of juice fit Mix A.

Going Further

Summarize

Revisit Questions A and B, and then discuss C and D. This summary should be a time for the students to give their ideas and to tell why they think their ideas make sense. You want the students to leave this problem having experienced several problem solving strategies, including an emphasis on ratios.

Suggested Questions

The main point that you want to be sure students think about is:

• What does it mean to be the most orangey tasting? To be the least orangey tasting?
Students might have a variety of ways to reason about the questions in this problem. One way to express the relationships of concentrate to total liquid in a batch is to write part-to-whole ratios. Then express the ratios as fractions and order the fractions (and thus the ratios) from least to greatest. Using prior knowledge about fractions, students may represent the fractions as decimals so that the comparisons are easy. From this, the students identify which is the most orangey (most concentrate, greatest fraction) and which is the least orangey (least concentrate, least fraction). Be sure that you ask the students to tell you what the fractions mean. The $\frac{1}{3}$ for Mix C means that one cup of juice has $\frac{1}{3}$ of a cup of concentrate and $\frac{2}{3}$ of a cup of water. Using this strategy, the order is from least to most orangey:

<table>
<thead>
<tr>
<th>Mix</th>
<th>Concentrate Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>B</td>
<td>$\frac{5}{14}$</td>
</tr>
<tr>
<td>D</td>
<td>$\frac{3}{8}$</td>
</tr>
<tr>
<td>A</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

You want to help the students see that they need to find a way to make something the same in order to make comparisons. Here are some possibilities that students have developed:

1. Make equal amounts of each mix and compare the amount of water or concentrate needed to make the chosen amount. For example, make 840 cups. Then we have 168 batches of Mix A; 60 batches of Mix B; 280 batches of Mix C; and 105 batches of Mix D. So the concentrate needed is 336 cups for Mix A; 300 cups for Mix B; 280 cups for Mix C; and 315 cups for Mix D. Mix A is most orangey and Mix C is least orangey.

2. Write a part-to-whole fraction from the amount of concentrate in each mix and the total amount of juice a batch of the mix makes. Find common denominators for these fractions to make comparisons. (Note that this is the same as making the same amount of each mix.) The fractions of concentrate would be $\frac{336}{840}$, $\frac{300}{840}$, $\frac{280}{840}$, and $\frac{315}{840}$.

3. Express the part-to-whole ratios as percents and compare. (This is in essence making the same amount of each mix.) The percents are 40%, 35.7%, 33.3%, and 37.5%.

4. Draw pictures showing how much water for each cup of concentrate is in each mix. (This makes the amount of concentrate, the denominator in the part-to-part fraction, the same.) The goal is to partition the water rectangles so that each cup of concentrate gets the same amount of water.

For Mix A, the middle rectangle is split into two equal parts and one part is given to each cup of concentrate.
So we have 1 ½ cups of water for each cup of concentrate. For Mix B, half the rectangles are split into five parts so that 1 whole and 4/5 of another can be given to each cup of concentrate.

![Diagram of water and concentrate for Mix B]

So we have 1 4/5 cups of water for each cup of concentrate. For Mix C, we give all of the water to the one concentrate and get 2 cups of water for each cup of concentrate.

![Diagram of water and concentrate for Mix C]

For Mix D, we have to partition two of the water into three parts so that we can share equally by giving 1 whole and 2/3 of another to each cup of concentrate.

![Diagram of water and concentrate for Mix D]

So we have 1 2/3 cups of water for each cup of concentrate.

5. Figure out how much water goes with each cup of concentrate. (This makes the denominators of the ratios the same.)

\[
\frac{1}{2}, \frac{4}{5}, \frac{2}{3}, \text{ and } \frac{1}{3}.
\]

Notice that this time we focus on the most and least watery. Most watery is 2/3.

If we figure out how much concentrate goes into each cup of water, we get

\[
\frac{2}{5}, \frac{5}{3}, \frac{1}{5}, \frac{3}{5}, \text{ where } \frac{1}{5} \text{ represents the least orangey.}
\]

6. Make the number of cups of concentrate the same.

Investigation 1
Other students might keep the number of cups of water the same and calculate the cups of concentrate.

7. Make a table to show the amounts of concentrate and water for each mix, and continue the values in the tables for multiples of the mixes until you have the same amount of water for each and can make a comparison.

When you hit 90 cups of water for each mix you will have: Mix A: 60 to 90; Mix B: 50 to 90; Mix C: 45 to 90; Mix D: 54 to 90.

Check for Understanding

* Which of the following will taste most orangey: 2 cups of concentrate and 3 cups of water; 4 cups of concentrate and 6 cups of water; or 10 cups of concentrate and 15 cups of water? (Research has shown that many students have preconceptions about the recipe scaling reasoning that will lead them to choose the recipe with most concentrate, even though the ratio of concentrate to water is the same in all three!)

Question C

Call on some students to explain their thinking. One example of a student response is: \( \frac{5}{9} \) is a part-to-part comparison and does not tell you what fraction of Mix B is concentrate. \( \frac{5}{14} \) is a part-to-whole comparison and does tell you what fraction of the mix is concentrate.

Question D

Again let groups report their results and give reasons why they think their results make sense. Here are four different ways of reasoning that groups have used with the different recipes.

Method 1

Some groups find the number of batches needed to make 120 cups of juice from each mix. Then they multiply to find the total amount of concentrate and the total amount of water. With Mix A, for example, one batch makes 5 cups of juice. We need 120, so we divide 120 by 5 to get the number of batches needed. We need 24 batches. Since we need 2 cups of concentrate for one batch, we need 2 x 24 cups of concentrate for 24 batches. This gives 48 cups of concentrate.

We need 3 cups of water per batch. So we need 3 x 24 cups of water for 24 batches. This is 72 cups of water. Since the amount of concentrate plus the amount of water equals 120 cups, we have a check on our work.

Method 2

Some groups use a special strategy for Mix C. They picture the 120 cups of water divided into three parts. Each part has 40 cups. We need 1 part concentrate, so this is 40 cups of concentrate. We need 2 parts water, so this is 80 cups of water.

Method 3

Some groups make a rate (ratio) table like the one here to help scale up.
Investigation 1

They recognize that the sum of 45 and 75 is the 120 that they need. This means 45 cups of concentrate and 75 cups of water. This strategy is clearly scaling ratios in the same way that students have found equivalent fractions.

Method 4

Some students use the strategy of finding equivalent fractions for the fraction of concentrate to the whole with 840 as the denominator, and then simplify to equivalent fractions with 120 as the denominator. Of course, Mix B makes 14 cups total, and since 14 is not a factor of 120, we cannot get a whole number of cups of concentrate.

Question E

This question is another way to focus students on the difference between part-to-whole and part-to-part comparisons. If you make exactly 1 cup of each mix, then you must deal with part-to-whole.

Focus Question

What strategies do you use to determine which mix is the most orangey?

Answers to Problem 1.2

A. Mix A will make the most orangey juice. (See possible explanations in the Summarize section.)
B. Mix C will make the least orangey juice. (See possible explanations in the Summarize section.)
C. 5/14 of Mix B is concentrate is correct because a part-to-whole comparison is needed to say the fraction part of the mix that is concentrate.
D. a. Mix A: Each 5-cup batch serves 10 people a half cup, so you need 24 batches. Mix B : Each 14-cup batch serves 28 people a half cup, so you need about 9 batches. Mix C: Each 3-cup batch

Investigation 1
serves 6 people a half cup, so you need 40 batches. Mix D: Each 8-cup batch serves 16 people a half cup, so you need 15 batches.

b. The table below shows how much of each ingredient is needed to serve 240 people. (See possible explanations in the Summarize section.)

<table>
<thead>
<tr>
<th></th>
<th>Mix A</th>
<th>Mix B</th>
<th>Mix C</th>
<th>Mix D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concentrate</td>
<td>24 x 2 cups</td>
<td>≈9 x 5 cups</td>
<td>40 x 1 cup</td>
<td>15 x 3 cups</td>
</tr>
<tr>
<td>Water</td>
<td>24 x 3 cups</td>
<td>≈9 x 9 cups</td>
<td>40 x 2 cups</td>
<td>15 x 5 cups</td>
</tr>
</tbody>
</table>

For the entries under Mix B, 43 and 77 are approximations. You might want to ask why you cannot make exactly 120 cups with whole numbers by scaling up this recipe?

E. Mix A would need \( \frac{2}{5} \) cup of concentrate and \( \frac{3}{5} \) cup of water. Mix B would need \( \frac{5}{14} \) cup of concentrate and \( \frac{9}{14} \) cup of water. Mix C would need \( \frac{1}{3} \) cup of concentrate and \( \frac{2}{3} \) cup of water. Mix D would need \( \frac{3}{8} \) cup of concentrate and \( \frac{5}{8} \) cup of water.

### 1.3 Time to Concentrate: Scaling Ratios

#### Introduction

**Description of Problem 1.3**

Problem 1.3 gives students the opportunity to practice the scaling strategies they have explored in 1.2. They choose part-part or part-whole ratios, as appropriate. The additional idea in 1.3 is that ratios in recipes can be re-stated as cans or cups or ounces, whatever is most convenient, as long as the order and the relative quantities are retained. That is, in the example of the lemonade, students may state the same ratio as 1 part concentrate: 5 1/3 parts mixture or, equivalently, 12 ounces concentrate: 5 1/3 × 12 ounces mixture. Part D pushes students to use the fraction form of ratios. Students have to provide the labels and make sense of the proportion formed. A scaling strategy will solve this proportion, foreshadowing 1.4, where proportions are defined and solving proportions is dealt with directly.

**Focus Question:**

*When you scale up a recipe and change the units, like from cups to ounces, what are some of the issues you have to deal with?*

#### Grouping

Small groups of 2 pairs will work well. There is often more than one way to do a problem, so pairs can work together and then compare their solutions with the other pair in the group, before sharing with the whole class.

#### Launch

*Connecting to Prior Knowledge*

### Investigation 1
Suggested Questions:

- When we scaled up ratios in the orange juice problem, to make comparisons, how do we know what each part of the scaled-up ratio means?
- What does part-to-part mean in terms of a recipe? Part-to-whole?
- The recipes in the orange juice problem are written in cans. Could we have written the recipes in ounces instead?
- Could we have used a fraction as a scale factor? What would be the result? Could we have used a mixed number as a scale factor?

Issuing the Challenge

What would you need to know about the recipe and the containers to decide which container (in the photograph) is best to use?

Explore

As you move around the room, listening to students discuss the Problem, these are some questions you might ask.

Suggested Questions:

- Is your recipe about part-to-part or part-to-whole? How did you decide?
- What is the unit measure in the parts of the ratio you wrote?
- Did you scale up a ratio? How did you decide what to multiply by? How do you know that the relationship between the ingredients is still correct?
- In part D, why are these two ratios equivalent?
- In part D, would scaling help? How can you decide on a scale factor to use?
- In part D, what does your answer mean?

Summarize

Much of Problem 1.3 is about trying out scaling strategies that will probably appear in the summary of Problem 1.2. Students will find out, however, that if they are not careful in setting up ratios, paying attention to units and order, so that meaning is retained, then scaling strategies will not work as intended. These common errors may appear in the Explore phase, and should be revisited in the Summary. Part D provides an opportunity to do all of this, and to begin to refine student scaling strategies.

Suggested Questions:

- In part D, what do the numbers mean in the ratios?
- In part D, how do we know that the numbers are in the right places, in numerators and denominators?
- In part D, why are these two ratios equivalent, even though they refer to different units?
- In part D, how might we scale up one ratio to find the missing number in the other ratio? Is there an efficient way to decide on the factor to use?
- In part D, how might we write the scaling up process, so it is clear what we did?

Investigation 1
Focus Question

When you scale up a recipe and change the units, like from cups to ounces, what are some of the issues you have to deal with?

Answers to Problem 1.3

A. The ratio of concentrate to mix is 1:4, so scaling this up we have 12 oz: 48 ounces. You need a container that holds 48 ounces. (This is a part-whole ratio.)

B. 1. The ratio of concentrate to mix is 1 to 5 1/3, so scaling this up we have 12 oz: 12 \times 5 \frac{1}{3} or 12 oz: 64 oz. So you need a container that will hold 64 oz.

2. ½ gallon container will hold this exactly. Students might prefer a container that is not full to the top.

C. 1 a. The ratio of concentrate to mix is 1:4, so scaling this up we have 16 oz: 16 \times 4 oz. We need a 64 ounce container.

b. One Gallon is 128 oz. Using the standard recipe the ratio of concentrate to mix is 1:4, so scaling this up we want to know X: 128. The scale factor is 32, so the ratio is 32:128. We need two 16 oz. Cans. (Students might use fractions, rather than ratios, arguing that the concentrate is ¼ of the whole mix, and ¼ of 128 is 32 ounces.)

2. August has 15 oz. of lemonade concentrate. The ratio of concentrate to mix is 1: 5 1/3, so scaling this up, with a factor of 15, we have 15 oz: 80 oz. A container that holds at least 80 ounces is needed. (Students might reason with cans instead of ounces. 1 can: 5 1/3 cans, scaled up, becomes 1 ½ cans: 5 1/3 \times 1 ½ cans or 1 ½ : 8 cans.)

D. 1. The 1 stands for the number of cans of concentrate that go into one recipe of orange juice. The 4 stands for the amount of juice made from one recipe, measured in cans. The 128 stands for the amount of juice Olivia plans to make, measured in ounces, so x stands for the number of ounces of concentrate Olivia needs.

2. There are a number of ways. One way is to think about these ratios as equivalent fractions. The denominator of the left hand fraction is multiplied by 32 to get the denominator of the right hand fraction. So, \( x = 32 \times 1 = 32 \).

1.4 Keeping Things in Proportion: Scaling to Solve Proportions

Introduction

Description of Problem 1.4

In this problem students encounter a variety of situations that require setting up proportions and solving for unknown values. Students should start to formalize the process of setting up equivalent ratios that make similar comparisons of quantities. In particular, scaling one ratio to find an unknown value in an equivalent ratio is highlighted as one strategy to help find unknown values.

Focus Question:

What strategies can you use to find a missing value in a proportion?
Investigation 1

Grouping:
Have students work in groups of 3 to 4.

Launch
Connecting to Prior knowledge
In the first three problems of Investigation 1, students wrote statements comparing quantities in various forms and made claims about these ratios compared to each other. In problem 1.3, students used a variety of strategies to scale ratios while solving contextual problems. In problem 1.4 students are formalizing what they learned in problem 1.3 by setting up a proportion and using the scaling-up or scaling-down strategy to find an unknown value.

Suggested Questions
Start the class by reminding students how Otis solved part D1 in problem 1.3. He set up the proportion \( \frac{1}{4} = \frac{x}{128} \) to represent the ratio of concentrate to orange juice. The \( \frac{1}{4} \) represents the given ratio (one can of concentrate to 4 cans of juice), and the \( \frac{x}{128} \) represents the ratio of some unknown value (x) of concentrate to 1 gallon = 128 oz. of orange juice.

Ask students the first question,
• Would it have made sense if Otis had written \( \frac{1}{x} = \frac{4}{128} \)?

The important thing that students should understand when setting up a proportion is that the ratios are in fact relating the same two quantities. For \( \frac{1}{4} \), this is a ratio comparing one can of concentrate in the standard recipe, to the unknown amount of concentrate in the scaled-up quantity (1-gallon jug). Similarly, the \( \frac{4}{128} \) represents a comparison of the total juice in the standard recipe to the scaled-up version. This means that Otis could have written the proportion this way. Students might also notice that in both cases x = 32 is a solution.

• What are some other ways that Otis might have written this proportion?
Using the same four values, students might re-write the proportions given by flipping both numerators and denominators, such as \( \frac{x}{1} = \frac{128}{4} \) or \( \frac{4}{x} = \frac{128}{1} \). If students happen to give correct proportions by comparing concentrate to water, that is fine, but you should point out that the comparisons are actually of different quantities than comparing concentrate to orange juice.

• Otis solved the proportion \( \frac{1}{4} = \frac{x}{128} \) by scaling up. He wrote \( \frac{1 \times 32}{x \times 32} = \frac{128}{1} \). How did he know to scale up by \( \frac{128}{1} \)?

Students may have different strategies including guess-and-check, however one important to strategy to make sure gets in the launch is that by looking at the denominator students are really trying to figure out what number multiplied by 4 will equal 128. Students can determine this missing value by solving \( 128 \div 4 = 32 \).

Note, this method works for both whole numbers and rational numbers. For example, if the proportion had been \( \frac{1}{4} = \frac{x}{128} \), you would still find the scale factor the same way \( 128 \div 4 = 31.5 \), resulting in a scale up by \( \frac{1}{4} = \frac{31.5}{1} \).

After discussing the problems related to Otis have students look at the four proportional statements comparing male to female doctors. Ask students how they would find the value for x. They may say that you can multiply the ratio \( \frac{5}{12} \) by \( \frac{50,000}{50,000} \). For the other question about
which proportion makes the most sense, there is no intended correct answer but if there are trends in your class about certain proportions you should make note of it. For example some students may prefer the x to be in the upper right position of the proportion, but it can be in any position in the proportion. If your students seem to quickly understand why all four proportions work you might consider putting up a proportion that is incorrect (e.g. comparing males to females and then females to males) and asking them if the proportion is correct or not and why. Articulating why it is incorrect may help some students understand that although there are many correct ways to set up a proportion not every single arrangement is correct.

Issuing the challenge

Tell students that they are going to be working on a variety of situations where they need to find unknown values. Emphasize that they should find the unknown values using strategies you have discussed as a class (such as using equivalent ratios, fractions, and scaling) and ones that make sense to them.

Note: you should consider how to structure questions D and E before having students work on the parts of problem 1.4. One possibility is to have students work through parts A-C and then use part D as a check for understanding during the summary portion because it starts to formalize the way to solve for an unknown in a proportion. In part D students are not given a context but instead asked to solve problems in a more abstract case. This question could be used to get a sense of how students are starting to generalize their strategies.

Part E could be used as a follow-up to part D, as a homework problem, or could precede part D if you wanted to use part D as an assessment type problem. You may also consider skipping part E completely if you feel that your students have a sound understanding of scaling equivalent ratios. Part E asks students to consider the strategy of finding a common denominator to solve a proportion. Sometimes, students prefer this strategy over scaling denominators to the other (for example when denominators are different prime numbers).

Note; this method explains why the cross-multiplication algorithm works because both fractions are being scaled to a like denominator (for fractions \( \frac{a}{b} = \frac{c}{d} \), they are scaled to \( bd \) so you can solve the equation with the numerators \( ad = bc \). The phrase “cross-multiplication” is not used in the student text here because often students are taught this method as a rote procedure without understanding. Students do not need to know the strategy (or even the term) of cross-multiplication. In many cases it confuses students and they can become reliant on using it without understanding why it works. The method of scaling should ground students’ solutions with more understanding of how the quantities are being operated on (including their units) when compared with using the cross-multiplication algorithm.

Explore

Suggested Questions

As students are working on questions A-D check to see if students are setting up the problems in a variety of ways. In the summary you might consider taking one specific problem (or all if you have time) and showing multiple ways of setting up the problem using proportions and equivalent ratios.

Some questions you might use to push students include:

• Are there other ways you could set up the relationship?
• Are there other ways to describe how the quantities compare?

Investigation 1
• Could the proportion be set-up differently?
• For part C, what connections could you make to what we did in Stretching and Shrinking?
• For part C, are the ratios you are writing part-to-part or part-to-whole?
• For part D, what strategies are you using to solve for x? Are there other ways of writing the same proportional situation?

Going Further

For part A, ask students if they have dogs and what dosage they should give. Or alternatively, if you have a dog you might give the approximate weight to students and see if they can come up with the correct dosage.

Up until part D, students have been writing proportional statements and then solving for the unknown value given a specific context. In part D, students are not given a context but just the bare numbers. A good extension is to have students consider different contexts that these proportions might model. For example, in D2, students might say that they can buy a dozen packs of sports cards for $7, how much would 9 packs cost? Encourage students to think about situations that are realistic given the values.

Summarize

Suggested Questions

Go over Question A asking groups to share their solutions and their strategies. One focus of the discussion should be on clear explanations for an algorithm (procedure) for solving a proportion for a missing quantity. Help students see that their knowledge of how to find equivalent fractions and ratios is very helpful here.

* I need a volunteer to give the answer to Question A and to explain how you found the answer and why it makes sense. (Repeat for all problems.)

* Does someone in another group want to challenge this solution or offer another solution strategy?

If this takes too long, wait until the solutions to all of the problems have been given and then reflect on the different strategies. Some students may comment that they prefer the way another group thought about the problem. If so, ask why—it may have been shorter, more efficient, more general, etc. Select different groups to represent different strategies that you observed during the exploration.

* If you had to solve a problem such as the ones in Problem 1.4 now, what solution strategy would you use and why?

Focus Question

What strategies can you use to find a missing value in a proportion?

Answers to Problem 1.4

A. Setting up the ratios as weight of dog in pounds: dosage in teaspoons, we have for Bruiser:
10: 1 = 80: x. We need to scale the original ratio by a factor of 8. So x = 8. Bruiser needs 8 teaspoons.

And for Dust Ball: 10:1 = 7:x. We need to scale the original ratio by a factor of 0.7. So x = 0.7. Dustball needs 0.7 teaspoons (about \(\frac{3}{4}\) of a teaspoon).

B. 1. Setting up the ratios as miles:calories, we have:

\[
\frac{5}{500} = \frac{x}{1200}
\]

\[x = 12\]

2. Setting up the ratios as miles:hours, we have:

\[
\frac{8}{2} = \frac{x}{x}
\]

\[x = 3\]

C. Because the triangles are similar, CE/BC = AE/AC, or \(\frac{h}{5} = \frac{48}{8}\). We could scale this by a factor of 0.625 and get \(\frac{48}{8} \times 0.625 = \frac{30}{5}\). So h = 30 ft. Or we could note that \(\frac{48}{8} = 6\), so we have to solve \(\frac{h}{5} = 6\), so h = 30.

D. 1. x = 20. The scale factor is 4, and 4 \(\times\) 5 = 20. \(\frac{7}{3} = \frac{32}{8}\) and \(\frac{8}{2} = \frac{32}{8}\).

2. x = 5.25. The scale factor is 0.75, and 7 \(\times\) 0.75 = 5.25.

\(\frac{7}{3} = \frac{4}{5}\) and \(\frac{32}{8} = \frac{4}{5}\). (Alternatively students might rename both ratios with a “denominator” of 36, so the proportion becomes 21/36 = 4x/36, so 4x = 21, so x = 5.25.)

3. x = 35. The scale factor is 5, and 5 \(\times\) 7 = 35.

\(\frac{5}{2} = \frac{7}{3}\) and \(\frac{5}{2} = \frac{7}{3}\).

4. x = \(\frac{8}{3}\). The scale factor is \(\frac{1}{4}\), and \(\frac{1}{4}\) of 8 = \(\frac{2}{3}\).

\(\frac{1}{4} = \frac{8}{3}\) and \(\frac{8}{3} = \frac{8}{3}\).

5. x = 200. Students might rewrite the proportion as x:5 = 40:1. The scale factor is 5, so x = 5 \(\times\) 40 = 200. (Or they might compare 5 and 3 to find the scale factor 5/3.)

6. x = 2/5. Students might rewrite the proportion as x:6 = 1:15, and then find a scale factor of 6/15 or 0.4. Or they might rewrite x/6 = 1/15 with common denominators of 30, so 5x/30 = 2/30, so 5x = 2, so x = 2/5 or 0.4. Or they might work with the original proportion, x/6 = 10/150, and find a scale factor of 6/150, so x = 10 \(\times\) (6/150) = 0.4.

E. 1. Because the denominators are now the same, 18 = 10x, so x = 1.8. (This strategy is equivalent to “cross multiplying” but makes more sense to students because it uses the same reasoning as they have used to scale equivalent ratios in other problems.)

2. Multiplying the ratio on the left by 3/3 and the ratio on the right by 5/5, Kevin’s proportion would be 9/30 = 5x/30. So 5x = 9, so x = 1.8.

Investigation 1
3. Multiplying the ratio on the left by 9/9 and the ratio on the right by 12/12, we get 63/108 = 12x/108, so 63 = 12x, so x = 5.25.

Applications, Connections and Extensions

Applications

Problem 1.1

1. In a comparison taste test of two drinks, 780 students preferred Berry Blast. Only 220 students preferred Melon Splash. Complete each statement.

   a. There were 560 more people who preferred Berry Blast.
   b. In the taste test 78% of the people preferred Berry Blast.
   c. People who preferred Berry Blast outnumbered those who preferred Melon Splash by a ratio of 39 to 11 (or 780 to 220)

Problem 1.1

2. In a comparison taste test of new ice creams invented at Moo University, 750 freshmen preferred Cranberry Bog ice cream while 1,250 freshmen preferred Coconut Orange ice cream. Complete each statement.

   a. The fraction of freshmen who preferred Cranberry Bog is 3/8.
   b. The percent of freshmen who preferred Coconut Orange is 62.5%; here students need to recognize that the fraction they need is 5/8, and 5 ÷ 8 = 0.625.
   c. The ratio of freshmen preferring Coconut Orange to those who preferred Cranberry Bog was 5 to 3

Problem 1.1

3. A town considers whether to put in curbs along the streets. The ratio of people who support putting in curbs to those who oppose it is 2 to 5.

   a. What fraction of the people oppose putting in curbs? 3/7
   b. If 210 people in the town are surveyed, how many do you expect to favor putting in curbs? 60
   c. What percent of the people oppose putting in curbs? About 71% (71.429%)
## Problem 1.1

Students at a middle school are asked to record how they spend their time from midnight on Friday to midnight on Sunday. Carlos records his data in the table below. Use the table for Exercises 4–7.

### Weekend Activities

<table>
<thead>
<tr>
<th>Activity</th>
<th>Number of Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sleeping</td>
<td>18</td>
</tr>
<tr>
<td>Eating</td>
<td>2.5</td>
</tr>
<tr>
<td>Recreation</td>
<td>8</td>
</tr>
<tr>
<td>Talking on the Phone</td>
<td>2</td>
</tr>
<tr>
<td>Watching Television</td>
<td>6</td>
</tr>
<tr>
<td>Doing Chores or Homework</td>
<td>2</td>
</tr>
<tr>
<td>Other</td>
<td>9.5</td>
</tr>
</tbody>
</table>

4. How would you compare how Carlos spent his time on various activities over the weekend? Explain.
   Possible answer: Fractions are a logical way to compare how students spent their time as they compare the time devoted to each activity (part) to the whole time investigated (whole).

5. Decide if each statement is an accurate description of how Carlos spent his time that weekend.
   a. He spent one sixth of his time watching television.
      No, \( \frac{6}{48} = \frac{1}{8} \)
   b. The ratio of hours spent watching television to hours spent doing chores or homework was 3 to 1.
      Yes, \( 6 : 2 = 3 : 1 \).
   c. Recreation, talking on the phone, and watching television took about 33\% of his time.
      \( 8 + 2 + 6 = 16 \)
      \( 16 \div 48 = 0.333 \approx 33\% \)
   d. Time spent doing chores or homework was only 20\% of the time spent watching television.
      No, \( \frac{2}{6} \approx 0.3333, 0.3333 \approx 33\% \approx 20\% \)
   e. Sleeping, eating, and “other” activities took up 12 hours more than all other activities combined.
      Yes, \( 18 + 2.5 + 9.5 = 30; 48 – 30 = 18; 30 – 18 = 12 \).

6. Estimate what the numbers of hours would be in your weekend activity table. Then write a ratio statement like statement 5b to fit your data.
   Answers may vary.

7. Write other accurate statements comparing Carlos’s use of weekend time for various activities. Use each concept at least once.
   a. ratio  
   b. difference  
   c. fraction  
   d. percent
   Possible answers:
a. The ratio of hours Carlos spent sleeping to hours he spent watching television is 3 to 1. The ratio of hours spent on the phone to doing chores or homework is 1 to 1.

b. The difference between the number of hours Carlos spent sleeping and the number he spent watching television is 12.

c. Carlos spent of his time on recreation.

d. Carlos spent 50% of his time watching television and sleeping. Carlos spent about 33% of his time on recreation, watching television, and doing chores and homework.

Problem 1.1

8. A class at Middlebury Middle School collected data on the kinds of movies students prefer. Complete each statement using the table.

<table>
<thead>
<tr>
<th>Types of Movies Preferred by Middlebury Students</th>
<th>Seventh-Graders</th>
<th>Eighth-Graders</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td>75</td>
<td>90</td>
</tr>
<tr>
<td>Comedy</td>
<td>105</td>
<td>150</td>
</tr>
<tr>
<td>Total</td>
<td>180</td>
<td>240</td>
</tr>
</tbody>
</table>

a. The ratio of seventh-graders who prefer comedies to eighth-graders who prefer comedies is [7:10]

b. The fraction of total students (both seventh- and eighth-graders) who prefer action movies is [11/28]

c. The fraction of seventh-graders who prefer action movies is [5/12]

d. The percent of total students who prefer comedies is [61%

e. The percent of eighth-graders who prefer action movies is [37.5%]

f. Grade [8] has the greater percent of students who prefer action movies.

Problem 1.1

9. In a survey, 100 students were asked if they prefer watching television or listening to the radio. The results show that 60 students prefer watching television while 40 prefer listening to the radio. Use each concept at least once to express student preferences.

a. ratio  b. percent  c. fraction  d. difference

Possible answers:

Investigation 1
Investigation 1

a. Students prefer radio to television by a ratio of 2 to 3 (2:3). Students prefer television to radio by a ratio of 3 to 2 (3:2).

b. 60% of students prefer television and 40% of students prefer radio.

c. \( \frac{3}{5} \) of students prefer television to radio.

d. The difference between the number of students who prefer television to radio is 20.
Problem 1.2

10. Compare these four mixes for apple juice.

![Mixes](image)

a. Which mix would make the most “appley” juice?

Mix Y is the most appley given it has the highest concentrate to juice ratio. The ratios of concentrate to juice are the following: Mix W = 5 : 13, Mix X = 3 : 9, Mix Y = 6 : 15, and Mix Z = 3 : 8. One possible strategy: If you changed these (part–whole ratios) to percents you will find Mix Y has the greatest percent of concentrate at 40%, whereas Mix W’s percent is about 38.5%, Mix X’s is about 33.3%, and Mix Z’s is 37.5%.

b. Suppose you make a single batch of each mix. What fraction of each batch is concentrate?

Mix W = \( \frac{5}{13} \)

Mix X = \( \frac{3}{9} = \frac{1}{3} \)

Mix Y = \( \frac{6}{15} = \frac{2}{5} \)

Mix Z = \( \frac{3}{8} \)

c. Rewrite your answers to part (b) as percents.

Mix W ≈ 38.5%, Mix X ≈ 33.3%, Mix Y = 40%, Mix Z = 37.5%

d. Suppose you make only 1 cup of Mix W. How much water and how much concentrate do you need?

Mix W: \( \frac{8}{13} \) cup water and \( \frac{5}{13} \) cup concentrate

Problem 1.2

11. Examine these statements about the apple juice mixes in Exercise 1. Decide whether each is accurate. Give reasons for your answers.

a. Mix Y has the most water per batch, so it will taste least “appley.”

Not accurate since both water and concentrate contribute to the least appley taste. A mix with 9 cups of water that had 1 cup of concentrate would taste much less appley.

b. Mix Z is the most “appley” because the difference between the concentrate and water is 2 cups. It is 3 cups for each of the others.

Not accurate. Mix Y is the most appley. Also, being the most appley is not dependent on the difference between the two ingredients, but the fraction or percent of concentrate of the total cups of liquid.

Investigation 1
c. Mix Y is the most “appley” because it has only 1 ½ cups of water for each cup of concentrate. The others have more water per cup.
   Accurate. Mix Y is the most appley because it has the greatest ratio of concentrate to water.

d. Mix X and Mix Y taste the same because you just add 3 cups of concentrate and 3 cups of water to turn Mix X into Mix Y.
   Not accurate. The taste is determined by the ratio of concentrate to water. Since Mix Y has more concentrate per water it will have the most appley taste.

Problem 1.2
12. If possible, change each comparison of concentrate to water into a ratio. If not possible, explain why.
   a. The mix is 60% concentrate.
      6 : 4
   b. The fraction of the mix that is water is \(\frac{3}{5}\).
      2 : 3
   c. The difference between the amount of concentrate and water is 4 cups.
      Not possible. This is discussing difference and to make a ratio, one would also have to know one of the amounts. Differences can be the same even when ratios between two quantities are different.

Investigation 1
Problem 1.3
A can of concentrated grapefruit juice has the instructions, “Mix one can of concentrate with 4 cans of cold water”. For exercises 1-6, use these mixing instructions to answer the questions.

13. Write a ratio for each of the situations, and then decide whether the situation is part-part or part-whole

   a. The ratio of water to concentrate
      4:1, part to part

   b. The ratio of concentrate to juice
      1:5, part to whole

   c. The ratio of water to juice
      4:5, part to whole

14. Which of the following ratios could represent this situation? If so, state what ratio it represents.

   a. $\frac{12}{60}$
      Ratio of concentrate to juice

   b. $\frac{3}{12}$
      Ratio of concentrate to water

   c. $\frac{2}{\frac{3}{5}}$
      Ratio of water to juice

   d. $\frac{5}{10}$
      This fraction does not represent a ratio from this situation

15. Orlando and Tanya are experimenting with different mix ratios. Determine if each situation will be a more concentrated (more “grapefruity”) or less concentrated (less “grapefruity”) than the original mix instructions.

   a. Mix A: 3 cans concentrate : 15 cans water
      Less concentrated (less grapefruity). Using the original mix and scaling by 3, 3 cans concentrate should be mixed with 12 cans of water. So, 15 cans of water makes the mix more watered down.

   b. Mix B: 3 cans concentrate: 15 cans juice
      The same concentration as the original. As stated in part a, 12 cans of water plus 3 cans of concentrate would give 15 cans of juice.

   c. Mix C: 10 cans cold water: 7 cans concentrate
      More concentrated (more grapefruity). Using the original mix and scaling by 2.5, 10 cans of water should be mixed with 2.5 cans of concentrate. Students might say the same concentration if they are mistakenly thinking of scaling as additive, that is, adding 6 cans of water and 6 cans of concentrate will give the same concentration. If students say this, you can point out that the ratio of concentrate to water in Mix C is over $\frac{3}{5}$, but the original is less than $\frac{3}{5}$.

   d. Mix D: $\frac{1}{4}$ can concentrate: 1 $\frac{1}{2}$ cans water
      Less concentrated (less grapefruity). Using a scale factor of $\frac{1}{4}$, $\frac{1}{4}$ can of concentrate should be mixed with 1 can of water.

Investigation 1
16. Jonathan and Samantha are making grapefruit juice from concentrate for a carnival. Jonathan mixes 10 cans of concentrate with 40 cans of water. Samantha mixes 8 cans of concentrate with 32 cans of water. Their teacher asks them to combine the two mixes into one large container. They are worried what will happen when they mix the two batches. What will happen to the new mixture?
   a) The new mixture will be less grapefruity.
   b) The new mixture will be the same as the original.
   c) The new mixture will be more grapefruity.

Explain your answer
   Answer B is correct. Because the original two mixtures were the same ratio as the original mixing instructions adding these two batches together will result in the same ratio. Interestingly, fractional notation may cause some difficulty if students consider this problem as \( \frac{10+8}{32+40} = \frac{1}{4} \).

17. Find the missing value for each situation
   a. 24 cans concentrate : x cans water
   Scale Factor is 24, 4 x 24 = 96 cans of water

   b. 24 cans concentrate : x cans juice
   Scale Factor is 6, 6 x 5 = 30 cans of juice

   c. 24 cans juice: x cans water
   Scale Factor is \( \frac{24}{5} \), 4.8 x 4 = 19.2 cans of water

   d. 24 cans juice: x cans concentrate
   Scale Factor is \( \frac{24}{5} \), 4.8 x 1 = 4.8 = \( \frac{4}{5} \) cans of water

18. Raina, Amelia, and Krista were trying to determine how many cans of concentrate would be needed if they filled 128 cans of water. They knew the problem they were trying to solve was \( \frac{1}{4} = \frac{x}{128} \). Which of the following strategies work? Explain.

   Raina’s strategy:
   I was looking for \( \frac{1}{4} \) of 128. I took 128 and divided it by four to find out what x equaled.
   This strategy works, 32:128 is equivalent to 1:4.

   Amelia’s strategy:
   I wrote a series of equivalent fractions by doubling the numerator and denominator.
   \( \frac{1}{4} = \frac{2}{8} = \frac{4}{16} = \frac{5}{32} = \frac{10}{64} = \frac{32}{128} \), so x = 32.
   This strategy works, 32:128 is equivalent to 1:4. Amelia is simply applying a scale factor of 2 at each step.

   Krista’s strategy:
   I factored the right side of the equation to determine x.
   \[
   \frac{1}{4} = \frac{x}{128} = \frac{1 \cdot 1 \cdot 2}{4 \cdot 4 \cdot 8}
   \]
   This strategy is incorrect. This fraction would be \( \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64} \neq \frac{1}{4} \)

Problem 1.4

19. Jared and Pedro walk 1 mile in about 15 minutes. They can keep up this pace for several hours.
a. About how far do they walk in 90 minutes?  
6 miles. Using equivalent ratios, $\frac{15}{1} = \frac{90}{?}$. The scale factor is 6.

b. About how far do they walk in 65 minutes?  
About 4.3 miles. The scale factor is $\frac{13}{5}$, or 4.333…

**Problem 1.4**

20. Swimming $\frac{1}{4}$ of a mile uses about the same number of Calories as running 1 mile.

a. Gilda ran a 26-mile marathon. About how far would her sister have to swim to use the same number of Calories Gilda used during the marathon?  
6.5 miles. Using equivalent ratios, $\frac{0.25}{1} = \frac{?}{26}$. The scale factor is 26.

b. Juan swims 5 miles a day. About how many miles would he have to run to use the same number of Calories used during his swim?  
20 miles. Using equivalent ratios, $\frac{0.25}{1} = \frac{5}{?}$. The scale factor is 20.

**Problem 1.4**

21. After testing many samples, an electric company determined that approximately 2 of every 1,000 light bulbs on the market are defective. Americans buy more than 1 billion light bulbs every year. Estimate how many of these bulbs are defective.  
About 2,000,000. Using equivalent fractions, $\frac{2}{1,000} = \frac{?}{1,000,000,000}$. The scale factor is 1 million.

**Problem 1.4**

22. The organizers of an environmental conference order buttons for the participants. They pay $18 for 12 dozen buttons. Write and solve proportions to answer each question. Assume that price is proportional to the size of the order.

a. How much do 4 dozen buttons cost?  
$\frac{18}{12\text{ dozen}} = \frac{?}{4\text{ dozen}}$. The scale factor is $\frac{1}{3}$. $18 \times \frac{1}{3} = 6$.

b. How much do 50 dozen buttons cost?  
$\frac{18}{12\text{ dozen}} = \frac{?}{50\text{ dozen}}$. The scale factor is $\frac{25}{6}$. $18 \times \frac{25}{6} = 75$.

c. How many dozens can the organizers buy for $27?  
18 dozen. $\frac{18}{12\text{ dozen}} = \frac{27}{?}$. The scale factor is $1.5$. $12 \times 1.5 = 18$.

d. How many dozens can the organizers buy for $63?  
42 dozen. $\frac{18}{12\text{ dozen}} = \frac{63}{?}$. The scale factor is $3.5$. $3.5 \times 12 = 42$.

**Problem 1.4**

Investigation 1
23. Denzel makes 10 of his first 15 shots in a basketball free-throw contest. His success rate stays about the same for his next 100 free throws. Write and solve a proportion to answer each part. Round to the nearest whole number. Start each part with the original 10 of 15 free throws.

   a. About how many free throws does Denzel make in his next 60 attempts?
      40. Using equivalent fractions, \( \frac{10}{15} = \frac{x}{60} \). The scale factor is 4.

   b. About how many free throws does he make in his next 80 attempts?
      About 53.3, or 53. The scale factor is about 5.3.

   c. About how many attempts does Denzel take to make 30 free throws?
      45 shots. Using equivalent fractions, \( \frac{10}{15} = \frac{30}{y} \). The scale factor is 3.

   d. About how many attempts does he take to make 45 free throws?
      About 68 shots. Using equivalent fractions, \( \frac{10}{15} = \frac{45}{z} \). The scale factor is 4.5.

Problem 1.4
For Exercises 24–30, solve each equation.

24. \( 12.5 = 0.8x \)
   \[ 15.625 = 12.5 \times 0.8 \]
   \[ x = 15.625 \]

25. \( \frac{x}{15} = \frac{20}{50} \)
   \[ 6 \times 15 = 20 \div 0.3 \times 20 = 6 \]

26. \( \frac{x}{18} = 4.5 \)
   \[ 81 \times 4.5 \times 18 = 81 \]

27. \( \frac{15.8}{x} = 0.7 \)
   \[ 22.6 \times 15.8 \div 0.7 = 22.6 \]

28. \( 245 = 0.25x \)
   \[ 980 \times 245 \div 0.25 = 980 \]

29. \( \frac{18}{x} = \frac{4.5}{1} \)
   \[ 4 \times 4.5 \div 18 = 0.25 ; 1 \div 0.25 = 4 \]

30. \( \frac{0.1}{48} = \frac{5}{960} \)
   \[ 2 \times 960 \div 48 = 20 ; 0.1 \times 20 = 2 \]

Problem 1.4
31. Multiple Choice Middletown sponsors a two-day conference for selected middle-school students to study government. There are three middle schools in Middletown.

Suppose 20 student delegates will attend the conference. Each school should be represented fairly in relation to its population. How many should be selected from each school?
Investigation 1

<table>
<thead>
<tr>
<th>North Middle School</th>
<th>Central Middle School</th>
<th>South Middle School</th>
</tr>
</thead>
<tbody>
<tr>
<td>618 Students</td>
<td>378 Students</td>
<td>204 Students</td>
</tr>
</tbody>
</table>

A. North: 10 delegates, Central: 8 delegates, South: 2 delegates

B. North: 11 delegates, Central: 7 delegates, South: 2 delegates

C. North: 6 delegates, Central: 3 delegates, South: 2 delegates

D. North: 10 delegates, Central: 6 delegates, South: 4 delegates

D. 10.3 (10) from North, 6.3 (6) from Central, and 3.4 (4) from South. The total from all schools is 1,200. The fraction of North to total is \( \frac{618}{1200} \), of Central is \( \frac{378}{1200} \), and of Central is \( \frac{204}{1200} \). Using equivalent fractions, \( \frac{618}{1200} = \frac{1}{2} \). The scale factor is \( \frac{1}{60} \). 618 x \( \frac{1}{60} = 10.3 \), 378 x \( \frac{1}{60} = 6.3 \), and 204 x \( \frac{1}{60} = 3.4 \). (Note: There may be alternative decisions to rounding up 6.3 instead of 3.4. In this instance, the decision was made based on the fact that 3.4 had a greater decimal value than 6.3.)

Connections

Problem 1.1

32. Suppose a news story reports, “A survey found that \( \frac{4}{7} \) of all Americans watched the Super Bowl on television.” Bishnu thinks this means the survey reached seven people and four of them watched the Super Bowl on television. Do you agree with him? If not, what does the statement mean?

No. It means that for every 7 people that responded to the survey, 4 of them watched the Super Bowl. If the survey only sampled 7 people, then 4 of them watched. But if the survey sampled 7,000 people, about 4,000 watched. \( \frac{4}{7} \) means that four sevenths of the number sampled, whatever that number is, watched. For example, if the survey sampled 100 people, \( \frac{4}{7} \times 100 \) would be 57.14.

Problem 1.1

33. Suppose a news story reports, “A survey found that \( \frac{4}{7} \) of all Americans watched the Super Bowl on television.” Bishnu thinks this means the survey reached seven people and four of them watched the Super Bowl on television. Do you agree with him? If not, what does the statement mean?

No. It means that for every 7 people that responded to the survey, 4 of them watched the Super Bowl. If the survey only sampled 7 people, then 4 of them watched. But if the survey sampled 7,000 people, about 4,000 watched. \( \frac{4}{7} \) means that four sevenths of the number sampled, whatever that number is, watched. For example, if the survey sampled 100 people, \( \frac{4}{7} \times 100 \) would be 57.14.

Problem 1.1

34. A fruit bar is 5 inches long. The bar will be split into two pieces. For each situation, find the lengths of the two pieces.
a. One piece is of the $\frac{3}{10}$ whole bar.
   One piece will be 1.5 in. and the other will be 3.5 in. A ratio of 3 : 7 also means that one piece will be 0.3 of the fruit bar and the other piece will be 0.7 of the fruit bar. Thus, $0.3 \times 5 = 1.5$ and $0.7 \times 5 = 3.5$.

b. One piece is 60% of the bar.
   One piece will be 3 in. long and the other will be 2 in. long (60% = 0.6, $0.6 \times 5 = 3$).

c. One piece is 1 inch longer than the other.
   One piece will be 3 in. long and the other will be 2 in. long.

Problem 1.1 # 15, 16 below could go with 1.4.
The sketches below show two members of the Grump family. The figures are geometrically similar. Use the figures for Exercises 13–16.

35. Write statements comparing the lengths of corresponding segments in the two Grump drawings. Use each concept at least once.
   a. ratio   b. fraction   c. percent   d. scale factor
   a. The ratio of the lengths of the top sides of the two Grumps is 0.8 to 1.2 or 2 to 3.
   b. Since they are similar, any side of the small Grump $\frac{2}{3}$ is the length of the larger Grump.
   c. The top side of the small Grump is about 67% of the length of the top side of the larger Grump.
   d. The scale factor from the small Grump to the large Grump is 1.5.

36. Write statements comparing the areas of the two Grump drawings. Use each concept at least once.
   a. ratio   b. fraction   c. percent   d. scale factor
   a. The ratio of the areas of the two Grumps is 4 to 9.
   b. The area of the smaller Grump is $\frac{4}{9}$ the area of the larger Grump.
   c. The area of the smaller Grump is about 44% the area of the larger Grump.

Investigation 1
d. The area scale factor from the small Grump to the large Grump is 2.25.

37. How long is the segment in the smaller Grump that corresponds to the 1.4-inch segment in the larger Grump?

0.93 in. (The scale factor is 1.5. Therefore, $1.4 \div 1.5 \approx 0.93$.)

38. **Multiple Choice** The mouth of the smaller Grump is 0.6 inches wide. How wide is the mouth of the larger Grump?
   A. 0.4 in.  B. 0.9 in.  C. 1 in.  D. 1.2 in.

B (0.6 times the scale factor of 1.5 equals 0.9.)

**Problem 1.2**

39. Exercise 11 includes several numbers or quantities: 5 inches, 3, 10, 60%, and 1 inch. Determine whether each number or quantity refers to the whole, a part, or the difference between two parts.

   The 3 in the numerator in part (a) and the 60% in part (b) each represent a part; the 5 inches in the problem text and the 10 in the denominator in part (a) represent a whole; and 1 inch in part (c) represents the difference between parts.

   **For the Teacher** Discuss what techniques were used by students to arrive at each of the answers. Which part was easiest to answer? Which way of phrasing the question (in terms of fractions, ratios, percents, difference) made the most sense for solving these problems?

**Problem 1.2**

40. If possible, change each comparison of red paint to white paint to a percent comparison. If it is not possible, explain why.

   a. The fraction of a mix that is red paint is $\frac{1}{4}$.
      25% red paint
   b. The ratio of red to white paint in a different mix is 2 to 5.
      28.6% red paint and 71.4% white paint

**Problem 1.2**

41. If possible, change each comparison to a fraction comparison. If it is not possible, explain why.

   a. The nut mix has 30% peanuts.
      $\frac{3}{10}$ peanuts
   b. The ratio of almonds to other nuts in the mix is 1 to 7.
      $\frac{1}{8}$ almonds and $\frac{7}{8}$ other nuts

**Problem 1.2**

42. Find a value that makes each sentence correct.

   a. $\frac{3}{15} = \frac{\_\_\_}{30}$
      6
   b. $\frac{1}{5} < \frac{\_\_\_}{20}$

Investigation 1
11, or any number greater than 10.

c. \( \frac{3}{4} \) or any number greater than 12.

d. \( \frac{3}{4} \) or any number greater than 20.

e. \( \frac{3}{4} \) or any number greater than 9.

f. \( \frac{3}{4} \) or any number greater than 28.

**Problem 1.2**

43. Use the table to answer parts (a) – (e).

<table>
<thead>
<tr>
<th>Participation in Walking for Exercise</th>
<th>Ages 12-17</th>
<th>Ages 55-64</th>
</tr>
</thead>
<tbody>
<tr>
<td>People Who Walk</td>
<td>3,781,000</td>
<td>8,694,000</td>
</tr>
<tr>
<td>Total in Group</td>
<td>23,241,000</td>
<td>22,662,000</td>
</tr>
</tbody>
</table>

a. What percent of the 55–64 age group walk for exercise?
   About 38.4%. (8,694,000 ÷ 22,662,000)

b. What percent of the 12–17 age group walk for exercise?
   About 16.3%. (3,781,000 ÷ 23,241,000)

c. Write a ratio statement to compare the number of 12- to 17-year-olds who walk to the number of 55- to 64-year-olds who walk. Use approximate numbers to simplify the ratio.
   The ratio of 12- to 17-year-olds who walk for exercise to 55- to 64-year-olds who walk for exercise is 3,781 to 8,694, or about 4 to 9.

d. Write a ratio statement to compare the percent of 12- to 17-year-olds who walk for exercise to the percent of 55- to 64-year-olds who walk for exercise.
   The ratio of the percentage of 12- to 17-year-olds who walk for exercise to the percent of 55- to 64-year-olds who walk for exercise is 8 to 19.

e. Which data—actual numbers of walkers or percents—would you use in comparing the popularity of exercise walking among various groups? Explain.
   Percents, because the number sampled in each category is not the same number, therefore percents seem more appropriate to use so that the two categories can be compared, based on numbers out of 100.

**Problem 1.3**

44. Copy the number line below. Add labels for 0.25, \( \frac{6}{8} \), 1 \( \frac{3}{4} \) and 1.3.

---

**Investigation 1**
Discuss how students are representing the numbers to place on the number line. Are they changing the way they are represented in the problem to a consistent means, such as all fractions or all decimals, etc.? What seems to be a natural way to begin dividing the segments on the number line?

Problem 1.3
45. Write two unequal fractions with different denominators. Which fraction is greater? Explain.
Possible answer: $\frac{3}{4} > \frac{2}{3}$, $\frac{3}{4}$ is greater because it is closer to 1. Its decimal equivalent is 0.75 as compared to about 0.67, the decimal approximation of $\frac{2}{3}$.

Problem 1.3
46. Write a fraction and a decimal so that the fraction is greater than the decimal. Explain.
Possible answer: $\frac{1}{2} > 0.25$ (0.5 > 0.25)

Copy each pair of numbers in Exercises 25–33. Insert $<$, $>$, or $\neq$ to make a true statement.

Problem 1.3 (1.1)
47. $\frac{4}{5} \quad \frac{11}{12}$
   $< \left(0.8 < 0.91 \text{ or } \frac{48}{48} < \frac{55}{55}\right)$
48. $\frac{14}{21} \quad \frac{10}{15}$
   $= \left(\frac{7}{3} = \frac{2}{3}\right)$
49. $\frac{7}{6} \quad \frac{4}{4}$
   $> \left(0.78 > 0.75 \text{ or } \frac{28}{36} < \frac{27}{36}\right)$
50. $2.5 \quad 0.259$
   $> \left(2.5 \text{ is greater than 1, and } 0.259 \text{ is less than 1}\right)$
51. $30.17 \quad 30.018$
   $> \left(\text{Because } 30 \text{ is the same in both, compare the tenths place; } 1 > 0, \text{ so } 30.17 > 30.018\right)$
52. $0.006 \quad 0.0060$
   $= \left(\text{Because the first three decimal places are the same in both, compare the next decimal place. The unwritten 0 in } 0.006 \text{ equals the 0 in } 0.0060, \text{ so } 0.006 = 0.0060\right)$
53. $0.45 \quad \frac{9}{20}$
   $= \left(0.45 = 0.45, \text{ or } \frac{9}{20} = \frac{9}{20}\right)$
54. $1 \frac{4}{4} \quad 1.5$
   $> \left(\frac{7}{4} > \frac{9}{4}, \text{ or } 1.75 > 1.5\right)$
55. $\frac{3}{4} \quad 1.3$
   $< \left(\frac{3}{4} \text{ is less than 1, and } 1.3 \text{ is greater than 1}\right)$

Investigation 1
Problem 1.3
56. Suppose a news story reports, “90% of the people in the Super Bowl stadium were between the ages of 25 and 55.” Alicia thinks this means only 100 people were in the stadium, and 90 of them were between 25 and 55 years of age. Do you agree with her? If not, what does the statement mean?

No. It means that 90%, or every 9 out of 10 people in the stadium, were between 25 and 55. There could have been 25,000 people in the stadium, in which case 22,500 would have been between the ages of 25 and 55 (25,000 \times 0.9 = 22,500). However, if there were only 100 people, then Alicia would be right that 90 were between those ages. Percents put actual numbers into a number that means “out of 100” in order to give a means of comparison.

Problem 1.3
57. Suppose a news story reports, “90% of the people in the Super Bowl stadium were between the ages of 25 and 55.” Alicia thinks this means only 100 people were in the stadium, and 90 of them were between 25 and 55 years of age. Do you agree with her? If not, what does the statement mean?

No. It means that 90%, or every 9 out of 10 people in the stadium, were between 25 and 55. There could have been 25,000 people in the stadium, in which case 22,500 would have been between the ages of 25 and 55 (25,000 \times 0.9 = 22,500). However, if there were only 100 people, then Alicia would be right that 90 were between those ages. Percents put actual numbers into a number that means “out of 100” in order to give a means of comparison.

Problem 1.4
58. Multiple Choice Choose the value that makes \( \frac{18}{30} = \frac{\_}{15} \) correct.

F. 7  G. 8  H. 9  J. 10

Answer: H

59. Multiple Choice Choose the value that makes \( \frac{\_}{15} \leq \frac{3}{5} \) correct.

A. 9  B. 10  C. 11  D. 12

Answer: A

60. Find a value that makes each sentence correct. Explain your reasoning in each case.

a. \( \frac{3}{4} = \frac{\_}{12} \)
   b. \( \frac{3}{4} < \frac{\_}{12} \)
   c. \( \frac{3}{4} > \frac{\_}{12} \)
   d. \( \frac{9}{12} = \frac{12}{\_} \)

   a. 9. The scale factor is 3 (12 ÷ 4 = 3 and 3 \times 3 = 9).
   b. 10. The numerator must be greater than 9 because \( \frac{9}{12} = \frac{3}{4} \).
   c. 8. \( \frac{9}{12} = \frac{3}{4} \) so the numerator must be less than 9.
   d. 16. The scale factor is \( \frac{4}{3} \) (12 ÷ 9 = \( \frac{4}{3} \) and \( \frac{4}{3} \times 12 = 16 \)).

Problem 1.4
61. Find values that make each sentence correct.

a. \( \frac{9}{14} = \frac{\_}{21} = \frac{\_}{28} \)
   \( \frac{9}{14} = \frac{9}{21} = \frac{12}{28} \)

b. \( \frac{\_}{27} = \frac{8}{36} = \frac{\_}{63} \)
   \( \frac{8}{27} = \frac{8}{36} = \frac{14}{63} \)

c. \( \frac{\_}{20} = \frac{\_}{25} = \frac{6}{30} \)
   \( \frac{4}{20} = \frac{5}{25} = \frac{6}{30} \)

d. \( \frac{28}{\_} = \frac{24}{32} \)

Investigation 1
\[
\frac{6}{8} = \frac{28}{37} \quad \text{(Note: the denominator here is 37 and one third) = } \frac{24}{32}
\]

**Problem 1.4**

62. Find a value that makes each sentence correct. Explain your reasoning in each case.

a. \( \frac{3}{4} = \frac{\_}{12} \)

b. \( \frac{3}{4} < \frac{\_}{12} \)

c. \( \frac{3}{4} > \frac{\_}{12} \)

d. \( \frac{9}{12} = \frac{\_}{12} \)

a. 9. The scale factor is 3 (12 ÷ 4 = 3 and 3 x 3 = 9).
b. 10. The numerator must be greater than 9 because \( \frac{9}{12} = \frac{3}{4} \).
c. 8. \( \frac{9}{12} = \frac{3}{4} \), so the numerator must be less than 9.
d. 16. The scale factor is \( \frac{9}{15} \) (12 ÷ 9 = \( \frac{4}{3} \) and \( \frac{4}{3} \) x 12 = 16).

**Problem 1.4**

63. **Multiple Choice** Ayanna is making a circular spinner to be used at the school carnival. She wants the spinner to be divided so that 30% of the area is blue, 20% is red, 15% is green, and 35% is yellow. Choose the spinner that fits the description.

Answer: B

**Problem 1.4**

64. Hannah is making her own circular spinner. She makes the ratio of green to yellow 2:1, the ratio of red to yellow 3:1, and the ratio of blue to green 2:1. Make a sketch of her spinner.

**Problem 1.4**

65. a. Plot the points (8, 6), (8, 22), and (24, 14) on grid paper. Connect them to form a triangle.

Investigation 1
b. Draw the triangle you get when you apply the rule \((0.5x, 0.5y)\) to the three points from part (a).

c. How are lengths of corresponding sides in the triangles from parts (a) and (b) related?
   The lengths are in proportion. The scale factor between the small triangle and the big triangle is 2 (or the scale factor between the large triangle and the small triangle is 0.5).

d. The area of the smaller triangle is what percent of the area of the larger triangle?
   The area of the small triangle is 25% of the area of the large triangle.

e. The area of the larger triangle is what percent of the area of the smaller triangle?
   The area of the large triangle is 400% of the area of the small triangle.

Problem 1.4
66. The sketch shows two similar polygons.
a. What is the length of side \(BC\)?

\[
BC \approx 3.42. \text{ Possible strategies: } \frac{6}{4} = \frac{3}{2} \\
BC = \frac{3}{2} \times 4 \approx 3.42, \frac{7}{6} = \frac{3}{2}. \text{ The scale factor is about } 0.57. 0.57 \times 6 = 3.42.
\]

b. What is the length of side \(RU\)?

\[
RU = 3.5. \text{ Possible strategies: } \frac{RU}{7} = \frac{2}{4} \\
RU = 7 \times \frac{2}{4} = 3.5. \frac{7}{2} = \frac{7}{RU}. \text{ The scale factor is } 1.75. 1.75 \times 2 = 3.5.
\]

c. What is the length of side \(CD\)?

\[
CD \approx 1.14. \text{ Possible strategies: } \frac{CD}{4} = \frac{2}{7} \\
CD = \frac{2}{7} \times 4 \approx 1.14. \text{ The scale factor is about } 0.57. 0.57 \times 2 = 1.14.
\]

Problem 1.4

67. To earn an Explorer Scout merit badge, Yoshi and Kai have the task of measuring the width of a river. Their report includes a diagram that shows their work.

![Diagram of a river with points A, B, C, D, and E, and measurements 300 m, 325 m, 650 m.](image)

a. How do you think they came up with the lengths of the segments \(AB\), \(BC\), and \(DE\)?

They most likely came up with the segments \(AB\), \(BC\), and \(DE\) by measuring. They could have used measuring tools or instruments to determine the length of segments they made. They probably staked off two points and measured the distance between them.

b. How can they find the width of the river from segments \(AB\), \(BC\), and \(DE\)?

Using equivalent ratios, based on the fact that the triangles are similar (Triangle \(ADE\) is similar to Triangle \(ABC\)). For example, if we compare corresponding sides of the large to the small triangle, we get \(650:325 = \frac{AD}{300}\) or \(\frac{650}{325} = \frac{AD}{300}\). Solving for \(AD\) gives \(AD = 600\). If we subtract \(AB\) from 600, we know that \(BD = 300\).
Extensions

Problem 1.1
68. Rewrite this ad so that it will be more effective.

Possible answer: About 67% of dentists recommend sugarless gum to their patients who chew gum. 2 out of 3 dentists recommend sugarless gum to their patients who chew gum.

Problem 1.1
69. Use the table below.

<table>
<thead>
<tr>
<th>Where Food Is Eaten</th>
<th>1990</th>
<th>1998</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home</td>
<td>$303,900,000,000</td>
<td>$401,800,000,000</td>
</tr>
<tr>
<td>Away From Home</td>
<td>$168,800,000,000</td>
<td>$354,400,000,000</td>
</tr>
</tbody>
</table>

a. Compare money spent on food eaten at home and food eaten away from home to the total money spent for food. Write statements for each year.

In 1990, about 64% of money spent on food was spent on food eaten at home. 36% was spent on food eaten away from home. (The total amount of money spent on food in 1990 was $472,700,000,000). In 1998, about 53% of money spent on food was for food eaten at home. 47% was spent on food eaten away from home.

b. Explain how the statements you wrote in part (a) show the money spent for food away from home increasing or decreasing in relation to the total spent for food.

The amount of money spent on food eaten away from home is increasing in relation to the total amount spent on food. 47% was spent on food eaten away from home in 1998 as compared to 36% in 1990.

Investigation 1
The two histograms below display information about gallons of water used per person in 24 households in a week.

**Problem 1.1 or 1.2, or 1.3 or 1.4 or Inv 2 or 3 (Inv 4)**

70. The two histograms below display information about gallons of water used per person in 24 households in a week.

![Histogram A: Water Use in 24 Households](image1)

![Histogram B: Water Use in 24 Households](image2)

**a.** Compare the two histograms and explain how they differ.

Histogram B uses larger intervals, so more households fit in each interval and the bars go higher. Histogram B is slightly more uniform on the lower end, while Histogram A overall contains more gaps and is not as uniform.

**b.** Where do the data seem to clump in Histograms A and B?

Possible answers: In Histogram A, the data seem to clump from 180 to 250 gallons, and in Histogram B, the data seem to clump from 160 to 260.

**Problem 1.1 or 1.3 CMP2 (Inv 1)**

Use the table for Exercises 36–41.
71. Which placement has the greatest difference in advertising dollars between 1990 and 2000?  
   television

72. Find the percent of advertising dollars spent for one type of placement in 1990.  
   See Figure 1

73. Find the percent of advertising dollars spent for one type of placement in 2000.  
   See Figure 1

74. Use your results from Exercises 36–38. Write several sentences describing how advertising spending changed from 1990 to 2000.  
   Possible answers:  
   Overall, the percent spent on advertising for each medium remains relatively consistent over the 10-year span from 1990 to 2000.  
   The percent spent on magazine advertising did not change over the 10 years.  
   The greatest difference in spending over the 10 years was in television.  
   The least difference in spending over the 10 years was in the Internet.  
   The greatest percent change in spending was in newspapers, down to 22% from 25%.
75. Suppose you were thinking about investing in either a television station or a radio station. Which method of comparing advertising costs (differences or percents) makes television seem like the better investment? Which makes radio seem like the better investment?

For television, discussing the difference makes television seem like a better investment because the percent of expenditures remained relatively consistent (22% as compared to 24%), yet the difference in actual dollar amount was 21,770,000,000. The difference in actual dollar amounts is therefore more impressive. The same is also true for radio as the difference between dollar expenditures would be impressive at 8,204,000,000 as opposed to the change in percents, from 7% in 1990 to 8% in 2000.

76. Suppose you are a reporter writing an article about trends in advertising over time. Which method of comparison would you choose?

Percents are easily understood and often used to discuss trends over time. In this case, they would indicate the relative consistency of expenditures per medium. The differences would highlight the impressive overall dollar increase in advertising. The differences would also make a better headline. However, the trends in advertising would be more accurately represented by using percents.

For the Teacher Discuss how such big differences can exist in terms of actual expenditures while percents can remain relatively unchanged.

1.4 CMP3 (Inv 4)
77. Angela, a biologist, spends summers on an island in Alaska. For several summers she studied puffins. Two summers ago, Angela captured, tagged, and released 20 puffins. This past summer, she captured 50 puffins and found that 2 of them were tagged. Using Angela’s findings, estimate the number of puffins on the island. Explain.

500. Using equivalent fractions, \( \frac{2}{50} = \frac{20}{?} \). The scale factor is 10.

Problem 1.4 (4.1)
78. Rita wants to estimate the number of beans in a large jar. She takes out 100 beans and marks them. Then she returns them to the jar and mixes them with the unmarked beans. She then gathers some data by taking a sample of beans from the jar. Use her data to predict the number of beans in the jar.

About 1,500 beans. Using equivalent fractions, \( \frac{5}{100} = \frac{50}{?} \). The scale factor is 50.

Investigation 1
Problem 1.4 or Inv. 2 or 3 (Inv 4)
79. The picture at the right is drawn on a centimeter grid.

a. On a grid made of larger squares than those shown here, draw a figure similar to this figure. What is the scale factor between the original figure and your drawing?

b. Draw another figure similar to this one, but use a grid made of smaller squares than those shown here. What is the scale factor between the original and your drawing?

One possible example is a picture that is either enlarged by a scale factor of 4 (going from the smaller figure to the larger figure), or reduced by a scale factor of (going from the larger figure to the smaller figure).

Note: Other scale factors could be used.

c. Compare the perimeters and areas of the original figure and its copies in each case (enlargement and reduction of the figure). Explain how these values relate to the scale factor in each case.

The perimeter of the similar figures can be found by multiplying the original scale factor by the corresponding scale factor of either the enlargement or the reduction. In the above example, the scale factor for the perimeter of the enlargement is 4 and the scale factor for the perimeter of the reduction is \( \frac{1}{4} \). The area of the two similar figures is found by multiplying the area of one figure by the square of the scale factor to determine the area of the other similar figure. In the example above, the scale factor for the area of the enlargement is \( 4^2 \) and the area for the reduced figure is \( \left( \frac{1}{4} \right)^2 \) or \( \frac{1}{16} \).

80. The people of the United States are represented in Congress, which is made up of the House of Representatives and the Senate.

a. In the House of Representatives, the number of representatives from each state varies. From what you know about Congress, how is the number of representatives from each state determined?
The number of representatives from each state is determined by the ratio of the population of the state to the population in the United States. Therefore, the greater the population of a state, the more representatives that state will have. Note: there is a minimum number of representatives so small states are still better represented proportionately than large states.

b. How is the number of senators from each state determined?

The number of senators is the same for every state, regardless of size or population. It is 2 per state.

c. Compare the two methods of determining representation in Congress. What are the advantages and disadvantages of these two forms of representation for states with large populations? How about for states with small populations?

With the same number for every state, small states can get an equal say/voice/vote, in terms of the Senate. However, with the method of the House of Representatives, the large states get more representation or voice, thus the Congress would be reflecting the voice of the people.

Mathematical Reflections

1. a. In this investigation you have used ratios, percents, fractions and differences to make comparison statements. How have you found these ideas helpful?

Students might explain how ratios are used to compare preferences, and that you can scale up from a sample to a larger population. They might say it is helpful to know what fraction or percent of a population prefers one outcome, so decisions are made to please the most people. The only time that difference is useful in making comparisons is when the absolute numbers are known. However, using the raw numbers to show a difference can impress if the difference is a large number.

b. Give examples to explain how part-to-part ratios are different from, but related to, part-to-whole ratios.

Students are likely to talk about recipes, where ingredients are the parts and the final mix is the whole. They should notice that they can use the part-to-part ratio to create the part-to-whole ratio. For example, if the part-to-part ratio is 2:3, then the related part-to-whole ratios are 2: (2 + 3) or 3: (2 + 3)

2. How do you use scaling or equivalent ratios

a. To solve a proportion? Give an example.

To solve, for example, \(5/3 = x/72\) students might scale up “3” by multiplying by 24, so 120/72 = x/72. Since these are equivalent we know \(x = 120\). It is important that students can describe how they decided that 24, in this example, was the necessary scale factor. (We need to multiply 3 by something to make 72, and 72 ÷ 3 = 24.)

To solve, for example, \(x:3 = 5:8\) students might rewrite both ratios as \(8x:24 = 15: 24\), so \(8x = 15\), so \(x = 15/8 = 1\frac{7}{8}\). (The common denominator variation of scaling up.)

Investigation 1
b. To make a decision? Give an example.

Students might talk about deciding how big a container is needed for mixing juice. For example, if a recipe calls for 1 can of concentrate to 3 cans of water, then the part-to-whole ratio is 1:4, and if the can of concentrate is 12 ounces then we need to solve the proportion 1:4 = 12: x, to decide how big a container we need.
Investigation 2 Comparing and Scaling Rates

Mathematical and Problem-Solving Goals

**Ratios, Rates and Percents:** Understand ratios, rates and percents.

- Use ratios, rates, fractions, differences, and percents to write statements in a given situation comparing two quantities
- Distinguish between and use both part-part and part-whole ratios in comparisons
- Use percents to express ratios and proportions
- **Recognize that rate is a special ratio that compares two measurements with different units.**
- Analyze comparison statements made about quantitative data for correctness and quality
- Make judgments about which statements are most informative or best reflect a particular point of view (for example, a percent and a fraction comparison or a difference and a ratio)

**Proportionality:** Represent and Recognize Proportionality in Tables, Graphs and Equations

- **Recognize that constant growth in a table or in a graph is related to proportional situations**
- Write an equation to represent the pattern in a table of proportionally related variables
- Connect a unit rate and constant of proportionality to the equation describing a proportional situation
- **Connect unit rate and constant of proportionality to a graph representing a proportional situation.**

**Reasoning Proportionally:** Strategies for Solving Problems

- Recognize situations in which proportional reasoning is appropriate to solve the problem
- **Scale a ratio, rate, percent, or fraction to make an equivalent ratio or rate (including unit rate), or to make a comparison.**
- Strategically find equivalent rates (including unit rates) and ratios to solve problems
- **Apply various strategies to solve for the unknown part when one part of two equal ratios is unknown, including scaling, rate tables, unit rates, equivalent ratios.**
- Set up and solve proportions that arise in applications for example, finding percents in the context of discounts and markups, converting measurement units.
Mathematical Goals and Mathematical Reflections

<table>
<thead>
<tr>
<th>Goals</th>
<th>Mathematical Reflections</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognize that constant growth in a table or in a graph is related to proportional situations</td>
<td>1. a. How do you find a unit rate or constant of proportionality in a table, a graph or an equation?</td>
</tr>
<tr>
<td>Connect a unit rate and constant of proportionality to the equation describing a proportional situation</td>
<td>b. When are tables, graphs and equations useful?</td>
</tr>
<tr>
<td>Connect unit rate and constant of proportionality to a graph representing a proportional situation.</td>
<td></td>
</tr>
<tr>
<td>Write an equation to represent the pattern in a table of proportionally related variables</td>
<td></td>
</tr>
<tr>
<td>Recognize that rate is a special ratio that compares two measurements with different units.</td>
<td>2. How are unit rates useful?</td>
</tr>
<tr>
<td>Scale a ratio, rate, percent, or fraction to make an equivalent ratio or rate (including unit rate), or to make a comparison.</td>
<td>3. How is finding a unit rate like solving a proportion?</td>
</tr>
<tr>
<td>Strategically find equivalent rates (including unit rates) and ratios to solve problems</td>
<td></td>
</tr>
<tr>
<td>Apply various strategies to solve for the unknown part when one part of two equal ratios is unknown, including scaling, rate tables, unit rates, equivalent ratios.</td>
<td></td>
</tr>
</tbody>
</table>

Teaching Notes for Each Problem

2.1 Sharing Pizza: Comparison Strategies

Introduction

2.1 Sharing Pizza: Comparison Strategies

Introduction

Description of Problem 2.1
In this problem, the question involves a comparison between two sizes of tables and two amounts of pizza. The question is whether the two situations are fair in terms of amount of pizza per person. The numbers are small, so that the idea of comparison can be the central focus. The problem is also designed to surface any wrong ideas students have about difference as a comparison strategy so that these can be challenged.

**Focus Question:**
What strategies can you use to compare two ratios to determine whether they are equivalent or determine which one is larger?

**Grouping:** Partners or groups work well for this problem

**Launch**

**Connecting to Prior knowledge**
In Investigation 1, students wrote comparison statements to make claims about whether ratios were equivalent, or which ratio was greater or smaller. They used strategies to re-write ratios in an equivalent form such as scaling up the ratio. In Problem 2.1, students apply what they learned in Investigation 1 by comparing two ratios and addressing which is larger or smaller.

**Suggested Questions**
To get the students engaged in the problem you might describe the situation and look at the two kinds of tables.

Ask, but do not analyze, whether each idea is a good one:

* At which table would you choose to sit?

* Do others have a different choice?

Talk together about what the problem is asking students to do so that they will have no problem interpreting the questions.

**Issuing the challenge**
Stress that each group should prepare to explain their strategy and why they think their strategy for solving the problem is appropriate. Also stress that you are very interested in the explanation of why students think their answers are correct. You should encourage the groups to try to find more than one way to make the comparison.

**Explore**

**Suggested Questions**
By this time, most of your students should find Question A of the problem very easy, although many of them may have difficulty discussing their solutions in the language of ratios. This is an example of a ratio that is related to quantities of different kinds—people and pizzas.

As you circulate, ask:

* Can you find another way to think about Question A? Are your two ways related? If so, how?

Question B requires students to figure out what another student has done and the claims she makes and then to decide whether her method is correct. The broader question in this part is whether analyzing differences is useful in determining who gets more pizza. Selena is comparing the difference of 6 to the difference of 5. Part B2 presses the issue more directly for students.

Investigation 2
Note, in part B1 if this was a valid comparison then it should follow that if there were 7 people with 1 pizza (a difference of 6) or 100 people with 106 pizzas (a difference of 6), that everyone should get the same amount which is clearly not the case in these extreme examples. If students are having difficulty with the values in question B you might suggest having them think about these extreme cases.

In question C students use what they learned in the previous investigation to set up equivalent ratios and solve for an unknown value. For example in C1, students should set-up the proportion $\frac{3}{8} = \frac{x}{160}$. Similarly in C2, the proportion should be $\frac{4}{10} = \frac{x}{160}$. For part C3, a fair number of pizzas means that students should get somewhere between $\frac{3}{8}$ and $\frac{4}{10}$ of a pizza. Students could set up a proportion and again solve for the unknown. For example $\frac{3}{8} = \frac{x}{25}$ or $\frac{4}{10} = \frac{x}{25}$. If students are having a difficult time with C3 ask them what the fraction $\frac{3}{8}$ stands for. It represents both how much of a pizza each person is getting, and the ratio of pizzas per number of seats at the table. This may help them write a proportion by making an alike comparison. In both cases, the number used to scale to a table of 25 is not a whole number. Students might have an easier time scaling $\frac{4}{10}$ by multiplying the numerator and denominator by 2.5, however starting with $\frac{3}{8}$ provides the opportunity to estimate, the scale factor is a little more than 3 (exactly 3.125).

**Going Further**

**Summarize**

**Suggested Questions**

Students will probably come up with a variety of ways of reasoning about this problem. Let them present to and quiz each other on their personally constructed approaches. For example, some students may say that they will join the large table, since it has more pizza, or the small table since it has fewer people to share pizza. In either case, using one number (the number of people or the number of pizzas) does not give enough information about the situation.

As students are sharing their strategies it is important to make clear (by either showing the students or pressing them to be clearer) about what the comparisons are in the ratios. It is important that when students use certain strategies such as scaling a ratio to an equivalent ratio you make note of that for the class explicitly. Looking ahead to Problem 2.2, students will be starting to use rate tables, so if a student creates a table of some sort that relates to that idea you might consider including this in the discussion so you can refer back to it in the next problem. If you include this, do not feel that you have to instruct students as to how to make it now; rather it is an opportunity for you to get a sense of how your students are making sense of a table to model a situation such as this.

Some questions to get the summary started include:

- **What if one table had 30 people and 5 pizzas and another had 5 people and 4 pizzas? Would you decide where to sit by choosing the table with the most pizza? Why or why not?**

- **What if one table had 10 people and 5 pizzas and another had 3 people and 1 pizza? Would you decide where to sit by just choosing the one with the fewest people? Why or why not?**

- **Does it make sense to add people and pizzas to make a whole population?** (No.)

There are two numbers associated with each table—the number of pizza and the number of people. Both of these values should be considered when the comparisons are made.

- **What is the ratio of pizza to people at the two tables?** (4 to 10 and 3 to 8)
• *What do these ratios mean?* (pizzas to people)

• *Can you scale these ratios so that they have a number in common?* (Yes. 4 to 10 and 16 to 40 are equivalent ratios; and 3 to 8 and 15 to 40 are equivalent ratios. This would mean 16 pizzas for 40 people compared to 15 pizzas for 40 people. Alternatively, a student might scale to 12 : 30 and 12 : 32. Note, scaling the numerators to equivalent values also allows for comparisons however in this context having the same number of pieces may make more sense.)

• *So, now which table gives you the most pizza?* (The 10-person table with 4 pizzas, which has a ratio equivalent to a 40-person table with 16 pizzas.)

• *Can you scale the ratios so that the number of pizzas is equal to make the comparison easy?* (Yes. 12 pizzas to 30 people is equivalent to the ratio of 4 pizzas for 10 people, and 12 pizzas for 32 people is equivalent to the ratio of 3 pizzas to 8 people. This means that you want to sit at the bigger table!)

Some students will compute unit rates which will show up later. This is fine, just be sure ratio arguments are also considered.

The large table has 4 pizzas to share with 10 people; this is $\frac{4}{10}$ pizza per person. At the smaller table, there are 3 pizzas to be shared with 8 people. Each person may have $\frac{3}{8}$ of a pizza.

• *If you want to go to the table with the most pizza per person, at which table do you sit? How do $\frac{4}{10}$ pizza and $\frac{3}{8}$ pizza compare?*

Students may want to look at equivalent fractions, decimals, or percents. If students look at the equivalent fractions, they could use 40 as a common unit so that $\frac{4}{10} = \frac{16}{40}$ and $\frac{3}{8} = \frac{15}{40}$. Because $\frac{16}{40}$ is greater than $\frac{15}{40}$, a student would get more pizza by sitting at a larger table. Some students will divide to write the fractions as decimals. This is fine. Just ask what labels the decimals should have.

Call on a couple of students to discuss Question B. The intent is to raise questions about difference as a way of making comparisons. Take the time to have the following discussion. You might assign these problems as part of the homework and discuss them the next day.

Let’s look at Selena’s reasoning in a fraction context: Selena compares fractions by subtracting the numerators and the denominators. She says the fraction with the lesser difference is closer to 1 and thus the greater fraction. Check Selena’s method on these fractions to see if it correctly predicts which is greater:

a. $\frac{3}{4}$ and $\frac{3}{5}$

b. $\frac{2}{7}$ and $\frac{3}{7}$

c. $\frac{4}{5}$ and $\frac{48}{50}$

What is interesting about Selena’s conjecture is that it is false in general but true for two important specific cases when the numerators are equivalent or the denominators are equivalent for fractions between 0 and 1.

• *What happens to a fraction when you add one to both the numerator and the denominator? Is the new fraction greater or lesser?*
In Question C, students are writing comparison statements, have students share their answers and ask them if there are other ways they could write their proportions (if they solved it that way). In particular C3 and C4 provide interesting opportunities for students to share their strategies of how they would create a fair number of pizzas to be placed on the table. Make sure that solution methods that include proportions get shared, but also make note of any other strategies that students might use to solve this problem.

**Focus Question**
What strategies can you use to compare two ratios to determine whether they are equivalent or determine which one is larger?

**Answers to Problem 2.1**

A. No. Because 4:10 and 16:40 are equivalent ratios; and 3:8 and 15:40 are equivalent ratios, we can see that sitting at the large table gives a person more pizza since 16 pizzas to 40 people is greater than 15 pizzas to 40 people.

(Students might make equivalent ratios with the same number of pizzas, say 12 pizzas: 30 people and 12 pizzas: 32 people. More people sharing the same number of pizzas gives each a smaller share.

Although rates are not defined until 2.2, and unit rates are not defined until 2.3, students might use a strategy that compares “pizza per person” or “person per pizza” for the two tables. They might reason that those at the large table will get \( \frac{4}{10} \), or \( \frac{2}{5} \), of a pizza per person. Those at the small table will get \( \frac{3}{8} \) of a pizza per person. Similarly, at the large table there are \( \frac{21}{2} \) people per pizza and at the small table there are \( \frac{22}{3} \) people per pizza. Each person at the large table will get more pizza than each person at the small table.)

B. 1. Some students will agree with Selena. They are still thinking additively. For them the comparison implied by a ratio is defined by the difference between the ratio parts. The next part of this question directly addresses this misconception.

2. Tony is correct. Placing 1 more pizza on the large table gives us a ratio of 5 pizzas to 10 people, or \( \frac{1}{2} \) pizza to one person. At the smaller table, sharing 3 pizzas among 8 people will give each of them less than \( \frac{1}{2} \) pizza – yet Selena’s focus on differences would incorrectly conclude that 5:10 and 3:8 should be the same.

C. 1. Using scaling as a strategy we have 3 pizzas: 8 people is equivalent to X pizzas: 160 people. The scale factor is 20, so we need \( 3 \times 20 = 60 \) pizzas. (20 is also the number of small tables needed.)

2. 4 pizzas : 10 people is equivalent to x pizzas : 160 people. Scale factor is 16, so \( x = 4 \times 16 = 64 \) pizzas. (16 is also the number of large tables needed.)
3. The camp director would probably try to make a ratio of pizzas to people for the extra large table, \( x \) pizzas: 25 people, that is equivalent to 3:8 (small table) or 4:10 (large table). 4:10 scales up with a factor of 2.5 to give 10 pizzas: 25 people. 3:8 scales up with a factor of 25/8 or 3.125, to give 9.375 pizzas: 25 people. (Students might also reason by comparing fractions of a pizza per person. The small table will allow 3/8 of a pizza per person; the large table will allow 4/10 of a pizza per person. The extra large table should allow a fraction of a pizza per person that is close to these fractions. Some students will find a fraction between these two fractions; between 30/80 and 32/80 is 31/80. If every person got 31/80 of a pizza we would need \( 25 \times 31/80 \), or between 9 and 10 pizzas for the extra large table.)

4. Depending on whether students choose to allow 9 or 10 pizzas per 25 people, the proportion is either \( x/160 = 9/25 \) (scale factor 6.4), or \( x/160 = 10/25 \) (scale factor 6.4). The problem with using the extra-large tables is that we cannot seat everyone at a “full” 25 person table. (The scale factor, 6.4, is the number of 25 person tables.) Students may ignore this and calculate that the director needs to buy 9 or 10 pizzas for each of 6.4 tables, resulting in 58, at 9 pizzas per table, or 64 pizzas, at 10 pizzas per table; or they may say that there will be 6 full tables (150 people), with 9 or 10 pizzas each, and one partial table (10 people) with 4 pizzas, resulting in 58 pizzas (at 9 pizzas per 25 people) or 64 pizzas (at 10 pizzas per 25 people).

2.2 Comparing Pizza Prices: Scaling Rates

Introduction

Description of Problem 2.2

In Problem 2.2 students systematically explore scaling ratios to solve problems. The ratios they work with in 2.2 are rates, that is, each ratio compares two quantities measured in different units, number of pizzas compared to price in dollars. This is not the first time students have worked with rates in Comparing and Scaling, but the first time the word has been explicitly defined. Students use a rate table to compare prices at two different pizzerias, taking advantage of patterns in the table to scale ratios up and down to find prices for other quantities of pizza, or to find the quantity of pizza that can be bought for a given price. They write equations relating the two variables. (The form of the equations, and the graphs, representing the relationship between total price and number of pizzas, foreshadows unit rate and constant of proportionality, ideas which are introduced in 2.3.) Part C contrasts two different situations, one of which is proportional and the other not.

Focus Questions:

_How do rate tables work to answer questions about different numbers or costs of pizzas?_  
_How are rate tables like scaling recipes and solving proportions?_

Grouping

Because students will have a variety of ways to work with the rate table, working in pairs will work well. They can share their strategies with the other pair before sharing with the whole class.

Launch

Connecting to Prior Knowledge

Suggested Questions:
• How might you write each pizzeria’s advertised prices as ratios?
• How might you compare the pizzerias’ prices?
• What else would you need to know before ordering from either of these pizzerias?
• If you know how much 10 pizzas cost, what costs for other numbers of pizzas can you quickly figure out? Can you write your thinking about this question as a proportion?

Issuing the Challenge

As you work on answering questions about costs of pizzas or numbers of pizzas, be on the lookout for connections with ratios and proportions.

Explore

As you move around the room, listening to students discuss the Problem, these are some questions you might ask.

Suggested Questions:

• What are some patterns you notice in the rate table in part A? Is there more than one way to find missing entries in the table?
• Is it easier to find the price for a certain number of pizzas, or to find the number of pizzas you can buy for a certain price? Or just the same?
• Can you find the price for any intermediate number of pizzas? Can you find the number of pizzas for any intermediate number of dollars?
• Is it easier to use the table or the equation to find the number of pizzas you can buy at Lion’s Den for $400? Would your answer be different if the number of dollars was different, say $140?
• How does the price for 1 pizza appear in the table, graph or equation? What about the price for 0 pizzas?
• When there is a no delivery charge can you double the cost of 5 pizzas to get the cost of 10 pizzas? Can you do this if there is a delivery charge?

Summarize

Students will have different strategies for finding the costs for different numbers of pizzas. One important strategy is to find the cost for 1 pizza, because all other costs can be figured from that. But it is important that other strategies appear; some students will double the cost for 10 pizzas to get the cost for 20 pizzas, some will halve the cost for 10 pizzas to get the cost for 5 pizzas, and then add the cost of 10 pizzas to the cost of 5 pizzas. This scaling-up strategy, and adding parts of two rates or ratios to create a new rate or ratio, only works for proportional relationships. It is important that students see how these strategies work, and also how they will not work in the situation where a delivery charge is added.

There are many connections among rate tables, proportions, equations and graphs in Problem 2.2. These should begin to come out in the summary. These connections are opportunities for students to use alternate ways to solve problems about proportional relationships.

Suggested Questions:

• One person doubled the cost of 10 pizzas at Lion’s Den to get the cost of 20 pizzas. Another person found the cost of 1 pizza at Lion’s Den and then multiplied by 20. Why do these give the same answer?

Investigation 2
• How is working with rate tables like scaling recipes? Could we have used a rate table to scale the orange juice recipes?

• How is working with rate tables like solving proportions? We could have written a proportion to answer a question like, “If 10 pizzas cost $120 at Lion’s Den, how many pizzas can we buy for $400?” How could we have solved this proportion? (The proportion is \( \frac{10}{120} = \frac{400}{n} \). Students might rewrite this as \( \frac{1}{12} = \frac{400}{n} \), and then scale up by multiplying 12 by 33 \( \frac{1}{3} \), to get \( \frac{33}{1} \times \frac{1}{12} = \frac{1}{3} \times 400 \), so \( n = 33 \frac{1}{3} \). This is essentially using the same unit price strategy as they used in the rate table. Or they may look for a common denominator for 120 and 400, and write \( \frac{400 \times 10}{120} = \frac{400 \times 400}{120} \), so 4000 = 120n, so \( n = 33 \frac{1}{3} \). This common denominator strategy does not have an immediate parallel in the rate table.)

• We can find the number of pizzas we can buy from Lion’s Den for $400 in the rate table, from an equation, or by setting up a proportion and solving. Which way do you prefer? Why?

• Why can you not double the cost of 1 pizza to get the cost of 2 pizzas, when there is a delivery charge?

• How does the cost for 1 pizza appear in the table? Graph? Equation? Would this be true for both Lion’s Den and Maverick? With and without delivery?

• How does the cost for 0 pizzas appear in the table? Graph? Equation? Would this be true for Lion’s Den and Maverick? With and without delivery?

Going Further

What is the equation for the price of pizzas at Maverick, with delivery?

What would the equation be if the delivery charge was $10 for any number of pizzas?

Would the equation \( C = 13(n + 5) \) fit the facts that 1 pizza costs $13 and the delivery charge is $5? Explain.

Focus Questions:

How do rate tables work to answer questions about different numbers or costs of pizza?

How are rate tables like scaling recipes and solving proportions?

Answers to Problem 2.2

A.

<table>
<thead>
<tr>
<th>Number of</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>100</th>
<th>150</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price $</td>
<td>12</td>
<td>24</td>
<td>36</td>
<td>48</td>
<td>60</td>
<td>120</td>
<td>180</td>
<td>240</td>
<td>1200</td>
<td>1800</td>
<td>2400</td>
</tr>
</tbody>
</table>

Investigation 2
Investigation 2

Price $  
Maverick Pizza  
| 13 | 26 | 39 | 52 | 65 | 130 | 195 | 260 | 1300 | 1950 | 2600 |

1. 53 pizzas from Lion's Den cost $636. One way to find this value from the table is to use half the price of 100 pizzas, and add the price of 3 pizzas. 27 pizzas from Maverick cost $351. One way to find this value from the table is to add the prices of 20, 5 and 2 pizzas.

2. With $400, they can buy 33 pizzas. One way to see this in the table is that 20 pizzas and 10 more would cost _______. This leaves $40 for more pizzas, which is enough for 3 pizzas, but not enough for 4 pizzas. Or students might work with a per pizza price. With $96 they can buy 8 pizzas (at $12 each).

B. 1. If you know the price of one pizza, you can multiply that price by any number of pizzas to find the total cost.

2. Using $P$ for total price and $n$ for number of pizzas, we can write these equations.

   Lion's Den: $P = 12n$ ($12$ is the price for 1 pizza at Lion’s Den.)
   Maverick: $P = 13n$. ($13$ is the price for 1 pizza at Maverick’s.)

3. Substituting $P = 400$ in the Lion’s Den equation gives $400 = 12n$. This equation represents the situation of knowing how much money we have to spend, $400$, and trying to find out how many pizzas we can buy, $n$. Solving this for $n$ gives $n = 400 ÷ 12 = 33 \frac{1}{3}$. (Of course, we cannot actually buy $1/3$ of a pizza.)

C. 1. a.

<table>
<thead>
<tr>
<th>N</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pizzas</td>
<td>18</td>
<td>31</td>
<td>44</td>
<td>57</td>
<td>70</td>
<td>135</td>
</tr>
<tr>
<td>D</td>
<td>13</td>
<td>26</td>
<td>39</td>
<td>52</td>
<td>65</td>
<td>130</td>
</tr>
<tr>
<td>Cost with delivery</td>
<td>Price if you pick up</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. The numbers in both lines of the table grow by 13 for every pizza added. The numbers in the “Delivery” line are each 5 more than the numbers in the “Pick up” line. The numbers in the “pick up” line can be scaled up to give other numbers in that line. The numbers in the “Delivery” line cannot be scaled up. (Note: This difference is crucial in recognizing from a
2. a.

b. The graphs are both lines, parallel to each other, but the graph representing the “with delivery” costs does not pass through (0, 0).

3. The table data values representing the “with delivery” charges does not fit the equation $P = 13n$. In fact it does not fit any equation in this format. (The equation is, in fact, $D = 13n + 5$. Students will encounter equations like this in later units.) The data values in the table for the pick up price can be found by using the equation from part B2.

### 2.3 Finding Costs: Unit Rate and Constant of Proportionality

**Introduction**

**Description of Problem 2.3**

In this problem, students work with unit rates to solve proportional relationship situations. By converting a given rate to a unit rate students are able to compare a one of the quantities in the ratio to a single unit of the other quantity. Converting to a unit rate is useful for scaling because you can simply multiply to get an equivalent ratio. One particular emphasis in problem 2.3 is finding unit rates through multiple relationships.

**Focus Question:**

Investigation 2
How do you find a unit rate when the rate is given in words, an equation, a table, or graph?

Grouping:
Have students work in groups of 3 to 4.

Launch through Part A

Connecting to Prior knowledge
In Problem 2.2 students worked with equations, rate tables, and graphs to determine the prices of a given amount of pizzas. This problem builds on their previous work with ratios and rates to help students create strategies for finding an important rate, the unit rate. Additionally, students are asked to reverse their thinking from the previous problem by asking students to interpret situations given in graphs, tables, and equation form to find the unit rate.

Note: this lesson may take longer than one day. The Launch-Explore-Summarize plan is given as a two day lesson, with the first day focused on part A and the second day on parts B and C. It may be that you will not need the whole class period the second day. You might consider doing the Mathematical Reflections for Investigation 2 as part of the second day’s lesson. If however, your students are able to get through part A quickly and you decide to do parts A, B and C in one day, you should still summarize after part A before continuing on to parts B and C.

Suggested Questions
Start class by reviewing some of the contexts in Investigation 1 and Investigation 2 where students seemed likely to use a unit rate, that is a comparison to one unit. The text in problem 2.3 refers explicitly back to problem 2.2, so you should include problem 2.2 as one example for sure. Having a larger set of examples however will be helpful for students to see the broader uses of unit rates in the unit up until this point. As one example, in part B of problem 1.1, students encountered a problem where ratio of students who preferred an athletic event to a concert was 2 to 1. In the Orange Juice Problem 1.2, some students may have found the orangiest juice by writing ratios compared to 1 cup of cold water, or 1 gallon of juice. In problem 2.2, students calculated the price for one pizza which is a good example of finding the unit rate for price per pizza. Providing these different examples (or even just one meaningful one) will help with the question,

• What is helpful about finding an equivalent ratio where the comparison is to one?

It may be helpful to put this in the context of the example you gave. If you chose Problem 2.2, you could re-state this as...

• Why was it helpful in Problem 2.2 for us to find the price for one pizza?

You can ask similar questions for the other contexts as well. One important reason that students should mention about problem 2.2 is that when you know the cost of one pizza you can easily calculate the cost for any amount of pizza by adding or multiplying. Tell students that we give this rate where the second quantity is 1, a special name, the unit rate.

Finding the unit rate is helpful for solving proportional relationships and is an important component of different representations of these relationships, such as equations, graphs, and tables.

Issuing the challenge
Display the context for part A of problem 2.3, that FreshFoods has oranges on sale 10 for $2. Ask students what unit rates they could determine for this situation (they could find the price for
1 orange, or they could find the number of oranges for $1.00). Tell students that they will be using unit rates to solve a series of problems on this context.

**Explore through Part A**

**Suggested Questions**

As students are working on part A focus students’ attention on the unit rate and constant of proportionality. For example if part A.4 you could find out what students are saying about how a unit rate allows you to find the cost for 25 oranges, or the number of oranges for 5 dollars. Students might say that by finding a unit rate that they can more easily determine the cost for any amount of oranges. Make note of these reasons for the summary portion.

In parts 5 and 6 students write equations and graphs. Some questions to ask students about the three representations (tables, equations, graphs) include:

- **If you had a graph how could you write an equation once you knew the constant of proportionality?** (If the constant of proportionality is \( p \), then the equation is \( y = px \) or for this context \( C = pn \)).

- **Where does the constant of proportionality appear in the table, graph, and equation?** (In the table it is the unit rate when one of the values equals one, on the graph it is the slope or the \( y \)-coordinate for the point \((1, y)\) or \((1,n)\), and in the equation it is the constant term \( p \) in the equation \( y = px \) or \( C = pn \)).

- **How does the table relate to the graph?** (All the solutions in the table are points on the graph)

**Going Further**

Ask students about what advantages or limitations they see in each representation. For example, what benefits are there in a graphical representation versus a table, or an equation versus an explanation in words.

**Summarize through Part A**

**Suggested Questions**

For parts 1-3, discuss with students how they calculated the cost per orange and the number of oranges they can buy for a dollar. Fill out the table in part 3.

In part 4 you should consider calling on students that have different reasons for why a unit rate might be helpful in finding the number of oranges you can buy for five dollars. For example one student might say that because the table is not well-ordered (the number of oranges row in the table does not increase by a constant amount) it would be helpful to know the number of oranges you could buy for one dollar. Another student might say that the number of oranges for $1 is easier than $2 (the given amount) because you can more easily multiply to get to $5 worth of oranges.

Discuss parts 5 and 6. The main goals of these two parts are to have students be able to identify the unit rate and constant of proportionality from an equation and graph. Additionally, you should press students to articulate what advantages they see in the different representations of the relationship between cost of oranges and number of oranges.

**Check for Understanding**

If you have time, there are two checks for understanding you might consider doing. As a first quick check you could put display various representations of proportional relationships and ask students to identify the unit rate or constant of proportionality. For example, display the equation \( C = 4n \), or show a table or graph similar to what was done in part A. This check is simply about identifying the constant of proportionality or unit rate given different representations. You might
also consider giving situations where they are not proportional, such as \( C = 4n + 2 \), to reinforce that proportional relationships are strictly multiplicative.

As a second check, you might give students a problem such as, *If two dozen apples cost $6, what is the unit rate? Write an equation, draw a graph, and create a table to show this relationship.* You could have students work on this if there is time, or you might talk through as a class how you would go about solving it. It may be interesting to hear which representation students would do first, some students may write the equation first, while others may be more comfortable with making the table or graph. Talking about the order may help students realize the advantages of starting with certain representations first and help their fluency in going between representations.

*Focus Question*

How do you find a unit rate when the rate is given in words, an equation, a table, or graph?

**Launch Parts B & C**

*Connecting to Prior knowledge*

In part A, students worked through the *FreshFoods* problem, figuring out the unit rate comparing the number of oranges to their cost. From this information, they filled out a table, wrote an equation, and created a graph. In parts B and C, students continue their work with unit rates but are asked to interpret situations given in graphical and equation form.

*Suggested Questions*

Start today’s class by reviewing the main points of part A.

- *We have been talking about rates and ratios for quite some time. Yesterday we talked about “unit rates”. What was special about a unit rate?* (It is a rate where one value in the ratio is one).
- *How was finding a unit rate helpful in the problem we worked on yesterday?* (By finding the unit rate it was easier to find the number of oranges for different dollar amounts)

For this last question you could have students share out as a whole class, or alternatively you might consider having students share with a partner or group members or simply write down some responses privately to get their thinking started about ways to find unit rates.

- *Yesterday we read that 10 oranges cost $2, and then we made a table, a graph and an equation. What were some strategies we used to find the unit rate in these different forms?*

If you do this as a whole class discussion make note of different strategies that students used. You might consider having these available for display for the class as students are working on parts B and C.

**Explore Parts B & C**

*Suggested Questions*

As you are monitoring students working through parts B and C, here are some questions you might ask.

For B1, students might be familiar with *miles per gallon* but have not considered *gallons per mile*. If this happens, remind students that *miles per gallon* describes the number of miles Noralie can drive on one gallon of gas, and then ask them if there is another way they could write the comparison.
In B2, students should recognize that the point (1,30) helps with finding the unit rate of 30 miles per one gallon. Make sure to pay attention carefully to how students are talking about this point on the graph. In general the point (1,y) helps with finding the unit rate but only if the relationship is proportional (that is the line goes through (0,0) as well).

In B4, you might ask students which of the two methods make sense to them and why. Both Josh and Lisa are correct, one uses a proportion the other converts the original relationship into a unit rate first.

In C3, students might give non-mathematical reasons for which store Gus should buy pasta at. That is certainly reasonable in real-life scenarios. You might re-phrase C3 as, “if it just came down to which one was the better deal, where should Gus buy the pasta?”

**Summarize Parts B & C**

*Suggested Questions*

Similar to the summary in part A, have students share their solutions and strategies to parts B and C highlighting how students used unit rates and the constant of proportionality in the different parts.

*Focus Question*

How do you find a unit rate when the rate is given in words, an equation, a table, or graph?

**Answers to Problem 2.3**

A. 1. $0.20

2. 5 oranges

3. 

<table>
<thead>
<tr>
<th>Number of oranges, ( n )</th>
<th>10</th>
<th>5</th>
<th>1</th>
<th>20</th>
<th>11</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ( C ), in dollars</td>
<td>2</td>
<td>1</td>
<td>0.20</td>
<td>4</td>
<td>2.20</td>
<td>2.60</td>
</tr>
</tbody>
</table>

4. If you know $1 buys 5 oranges then $5 buys five times as many, so 25 oranges. If you know 1 orange costs $0.20 then 25 oranges cost 25 times as much, so $5.

5. 

a. The equation shows that the number of oranges is \( 5 \times \) number of dollars spent.

b. \( C = 0.2n \). (or \( C = n/5 \)) This equation says that the cost \( C \), in dollars is 1/5 of the number of oranges.

c. The unit rate is the multiplier or coefficient of the independent variable.

d. The constant of proportionality is equal to the unit rate in each case.

6. a.
b. The unit rates are given by the points $(1, X)$. In the case of $n = 5C$ then $(1, 5)$ is on the graph, and the unit rate is 5 oranges per 1 dollar. In the case of $C = 0.2n$ then $(1, 0.2)$ is on the graph, and the unit rate is $0.20$ per 1 orange.

c. Since constants of proportionality and unit rate are equal, see the answer above. (Note: The constant of proportionality and unit rate are measures of the slope, $m$, of the line $y = mx$, which students will study in *Moving Straight Ahead*.)

B. 1. Students might make rate tables, or proportions.
600 miles : 20 gallons = D miles: 1 gallon. D = 30 miles. Unit rate is 30 miles per gallon. It tells you that for every 30 miles driven you use 1 gallon. (You might use this if you want to know how many miles you can go on a full tank of gas, say 15 gallons. (15 × 30 miles.))

600 miles: 20 gallons = 1 mile : G gallons. G = 20/600. Unit rate is 1/30 gallon per 1 mile. It tells you that for every 1/30 of a gallon of gas used you travel 1 mile. (You might use this if you want to know how many gallons it will take to make a journey of 200 miles. (200 × 1/30 gallons))

2. The unit rate on the graph is 30 miles per 1 gallon. You can see this on the graph at (1, 30).

3. D = 30g

4. a. They are both correct. Josh’s proportion compares miles: gallons. 600 miles: 20 gallons = x miles: 4 gallons. (The scale factor is 1/5, so x = 120 miles.) If Lisa uses the unit rate 30 miles per 1 gallon of gas, she can compute the number of miles for 4 gallons of gas. (4 × 30).

b. Students might make a rate table, or a graph, or use the equation D = 30g.

C. 1. The first proportion says “if 7 boxes cost $6 then n boxes would cost $1.” So n = 7/6 is a unit rate telling us that we get 7/6 or 1.1666 boxes for $1. The second proportion says “if $6 is the cost for 7 boxes then C is the cost for 1 box.” So 6/7 is a unit rate telling us that $0.86 is the cost for 1 box.

2. The rate 6 boxes/$5 gives us the unit rate 1.2 boxes for $1. The rate $5 for 6 boxes gives us the unit rate $0.83 for 1 box.

3. Gus should buy at FreshFoods because 1 box costs $0.83 at FreshFoods and $0.86 at More For Your Money. (Or students might compare the unit rates as fractions, 5/6 < 6/7.)

Applications, Connections and Extensions

Applications

Problem 2.1

1. Guests at a pizza party are seated at 3 tables. The small table has 5 seats and 2 pizzas. The medium table has 7 seats and 3 pizzas. The large table has 12 seats and 5 pizzas. The pizzas at each table are shared equally. At which table does a guest get the most pizza?

The medium table; at the medium table, each person gets about \( \frac{3}{7} \), or 43%, of a pizza. In other words, there are about 2.3 people per pizza. At the small table, each person gets only, or 40%, of a pizza. There are 2.5 people per pizza. At the large table, each person gets about \( \frac{5}{12} \), or 42%, of a pizza. There are 2.4 people per pizza.

Problem 2.1

2. Suppose a news story about the Super Bowl claims “Men outnumbered women in the stadium by a ratio of 9 to 5.” Does this mean that there were 14 people in the stadium—9 men and 5 women? If not, what does the statement mean?
No, but if there had been only 14 people, then 9 would have been male and 5 would have been female. It means for every 9 men in the entire stadium, there were 5 females. So if there were 9,000 males, there were 5,000 females.

**Problem 2.1**

3. **Multiple Choice** Which of the following is a correct interpretation of the statement “Men outnumbered women by a ratio of 9 to 5?”

A. There were four more men than women.
B. The number of men was 1.8 times the number of women
C. The number of men divided by the number of women was equal to the quotient of 5 x 9.
D. In the stadium, five out of nine fans were women.

Correct Answer is B

**Problem 2.2**

4. For each business day, news reports tell the number of stocks that gained (went up in price) and the number that declined (went down in price). In each of the following pairs of reports, determine which ratio shows the larger gain.

a. Gains outnumber declines by a ratio of 5 to 3 OR Gains outnumber declines by a ratio of 7 to 5.
   The ratio of 5 to 3 is better than 7 to 5. In the ratio of 5 to 3, 5 out of every 8 people (0.625 or 62.5%) gain whereas with the ratio 7 to 5, 7 out of every 12 people (0.58333 or 58.3%) gain. Another way to look at it is the ratio of 5 : 3 = 1.6667 and the ratio 7 : 5 = 1.4.

b. Gains outnumber declines by a ratio of 9 to 5 OR Gains outnumber declines by a ratio of 6 to 3.
   The ratio of 6 : 3 is better than 9 : 5. ($\frac{6}{9} > \frac{9}{14}$, 67% > 64%).

c. Gains outnumber declines by a ratio of 10 to 7 OR Gains outnumber declines by a ratio of 6 to 4.
   The ratio of 6 to 4 is better for investors. $\frac{6}{10} = 60\%$ whereas $\frac{10}{17} \approx 58.8\%$

**Problem 2.2**

The problems that follow will give you practice in using rates (especially unit rates) in different situations. Be careful to use measurement units that match correctly in the rates you compute.

5. Maralah can drive her car 580 miles at a steady speed using 20 gallons of gasoline. Make a rate table showing the number of miles her car can be driven at this speed. Show 1, 2, 3, . . . , and 10 gallons of gas.

Maralah’s Driving Distance
Problem 2.2
6. Joel can drive his car 450 miles at a steady speed using 15 gallons of gasoline. Make a rate table showing the number of miles his car can be driven at this speed. Show 1, 2, 3, . . . , and 10 gallons of gas.

<table>
<thead>
<tr>
<th>Gallons</th>
<th>Miles Driven</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>58</td>
</tr>
<tr>
<td>3</td>
<td>87</td>
</tr>
<tr>
<td>4</td>
<td>116</td>
</tr>
<tr>
<td>5</td>
<td>145</td>
</tr>
<tr>
<td>6</td>
<td>174</td>
</tr>
<tr>
<td>7</td>
<td>203</td>
</tr>
<tr>
<td>8</td>
<td>232</td>
</tr>
<tr>
<td>9</td>
<td>261</td>
</tr>
<tr>
<td>10</td>
<td>290</td>
</tr>
</tbody>
</table>

Problem 2.2
7. Franky’s Trail Mix Factory gives customers the following information. Use the pattern in the table to answer the questions.

<table>
<thead>
<tr>
<th>Gallons</th>
<th>Miles Driven</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>120</td>
</tr>
<tr>
<td>5</td>
<td>150</td>
</tr>
<tr>
<td>6</td>
<td>180</td>
</tr>
<tr>
<td>7</td>
<td>210</td>
</tr>
<tr>
<td>8</td>
<td>240</td>
</tr>
<tr>
<td>9</td>
<td>270</td>
</tr>
<tr>
<td>10</td>
<td>300</td>
</tr>
</tbody>
</table>
a. Fiona eats 75 grams of trail mix. How many Calories does she eat?

225 Calories. Since 150 g of trail mix contains 450 Calories, an equivalent ratio of grams to Calories is 1 : 3. From this scaled-down ratio, you can scale up to 75:225, which means that 75 g of trail mix would contain 225 Calories.

b. Rico eats trail mix containing 1,000 Calories. How many grams of trail mix does he eat?

333.33 . . . (≈ 333). The ratio of Calories to grams is 3 to 1. 1,000:333.33 . . . is equivalent. Or, 1,000 Calories is 2/3 of 1,500 Calories, so Freddy ate of 500 g, or about 333 g.

c. Write an equation that you can use to find the number of Calories in any number of grams of trail mix.

Number of Calories = 3 x number of grams (C = 3g)

d. Write an equation that you can use to find the number of grams of trail mix that will provide any given number of Calories.

Number of grams = number of Calories ÷ 3 (g = C ÷ 3, or g = C/3)

Problem 2.2

8. At camp, Miriam uses a pottery wheel to make three bowls in 2 hours. Duane makes five bowls in 3 hours.

a. Who makes bowls faster, Miriam or Duane?

Duane. He can make about 1.7 (5 ÷ 3) bowls per hour and Miriam can make only 1.5 bowls per hour.

b. At the same pace, how long will it take Miriam to make a set of 12 bowls?

8 hours because 2/3 = 8/12

c. At the same pace, how long will it take Duane to make a set of 12 bowls?

It will take Duane a little over 7 hours, or about 7.2 hours to make 12 bowls. Possible strategy: 5 ÷ 3 = 1 2/3 and 12 ÷ 1 2/3 = 7.2.

Problem 2.3 (3.1)

9. The dairy store says it takes 50 pounds of milk to make 5 pounds of cheddar cheese.

a. Make a rate table showing the amount of milk needed to make 5, 10, 15, 20, . . . , and 50 pounds of cheddar cheese.
b. Make a coordinate graph showing the relationship between pounds of milk and pounds of cheddar cheese. First, decide which variable should go on each axis.

c. Write an equation relating pounds of milk \( m \) to pounds of cheddar cheese \( c \).
\[
\frac{1}{10} m = c, \text{ or } m = 10c
\]

d. What is the constant of proportionality in your equation from part c?
\[
\frac{1}{10} \text{ for the equation } \frac{1}{10}m = c
\]
\[
10 \text{ for the equation } m = 10c
\]

e. Explain one advantage of each method (the graph, the table, and the equation) to express the relationship between milk and cheddar cheese production.
Possible answers: The graph visually shows the relationship between amounts of milk and cheese. The table allows one to look up how much milk is needed to yield any given cheese amount. The equation allows for quick calculation of the amount of milk needed for any amount of cheese.

Problem 2.3
10. Keeley is downloading songs from a new music website. She buys 35 songs for $26.25
a. What is the price per song?
\[
$0.75
\]
b. Alison gets a $50 gift card for the music site. She’s trying to estimate how many songs she could buy using the gift card. Which estimate seems the most reasonable to you. Explain.
   
   i) Somewhere between 30 and 50 songs
   ii) Around 70 songs, but definitely less than 70.
   iii) Around 70 songs, but definitely more than 70.
   iv) For sure at least 90 songs.

   Because 35 songs is $26.25 = $25.00, 70 songs would be around $50.00; however since $26.25 > $25.00, it would less a little less than 70 songs. The correct answer is ii.

   c. Complete the rate table below

<table>
<thead>
<tr>
<th>Number of songs, $n$</th>
<th>35</th>
<th></th>
<th>50</th>
<th>1</th>
<th>70</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost in dollars, $C$</td>
<td>$26.25</td>
<td>3</td>
<td></td>
<td>?</td>
<td>?</td>
<td>$15.00</td>
</tr>
</tbody>
</table>

   Answers:

<table>
<thead>
<tr>
<th>Number of songs, $n$</th>
<th>35</th>
<th>4</th>
<th>50</th>
<th>1</th>
<th>70</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost in dollars, $C$</td>
<td>$26.25</td>
<td>$3.00</td>
<td>$37.50</td>
<td>$0.75</td>
<td>$53.50</td>
<td>$15.00</td>
</tr>
</tbody>
</table>

   d. Benjamin and Franklin are discussing how to write an equation for this situation. Benjamin says that the equation should be: $n = .75C$, but Franklin says the equation should be $C = .75n$. How could you use the information from parts a – c, to convince yourself of which equation is correct?

   Franklin is correct. One way you could use the information is to substitute the original number of songs ($n = 35$) into both equations, and see which one gives the cost in dollars. Benjamin’s equation produces a cost of approximately $19.69, which is incorrect, but Franklin’s equation gives $26.25 which is correct. Note: in this case it is assumed that one of the two equations is correct, however students should get in the habit of thinking about what the equation is “saying” in general and not just checking one value. Franklin’s equation is saying that the cost “$C” is equal to the number of songs multiplied by $0.75 per song, which is the relationship in this situation.

Problem 2.3

11. Determine a unit rate for each situation, then write an equation relating the two quantities.

   a. 3 dozen apples for $4.50
      $1.50 per dozen, or about $0.13 per apple
      $C = $1.50d

   b. 30-pack of bottled water for $4.80
      $0.16 per bottle
      $C = $0.16b

   c. 24 ounces of mozzarella cheese for $2.88
      $0.12 per ounce
      $C = $0.12m

Problem 2.3
For exercises 4-5, Courtney notices that the Back to School prices for different school supplies seems cheaper than the regular price the rest of the year. However, the Back to School supplies requires buying “in bulk” (more than just a single item), so she’s not sure whether she is actually getting a good deal or not.

12. Which of these items is the better buy?

   a) An 8-pack of glue sticks for $3.99 or 1 glue stick for $0.54  
      The 8-pack is the better deal, each glue stick is around $0.50.

   b) A 12-pack of tape for $2.50 or 1 roll of tape for $0.19  
      The single roll is the better deal, each roll in the 12-pack is around, but greater than $0.20.

   c) A 100-pack of pencils for $4.88 or 1 pencil for $0.05  
      The 100-pack is the better deal, 100 pencils for $0.05 a piece would cost $5.00.

   d) 40 pencil-top erasers for $2.82 or a 2-pack of pencil-top erasers for $0.12.  
      Buying the two-packs is cheaper, twenty 2-packs (40 total) would cost $0.12 x 20 = $2.40.

Problem 2.3

13. Courtney’s parents were not convinced of her answer in part d, so she tried to explain it in a lot of different ways. Which of these methods are correct ways to get the solution for part d? Which is the most convincing to you?

Method 1:  
Compare the two unit-rates to determine which unit rate is cheaper.  
\[
\frac{2.82}{40} = \frac{x}{1} \Rightarrow x = 0.0705 \approx $0.07 \text{ per eraser}
\]

\[
\frac{0.12}{2} = \frac{x}{1} \Rightarrow x = 0.06 = $0.06 \text{ per eraser}
\]

The price per eraser is cheaper using the two-packs.

Method 2:  
If I buy forty of the smaller packs, that will be 40 x $0.12 = $4.80 which is more expensive than $2.82 for forty erasers in the main pack. The 40-pack is the better deal.

Method 3:  
If a 2-pack costs $0.12, then twenty 2-packs would be 40 pencil-top erasers. Twenty 2-packs cost 20 x $0.12 = $2.40 < $2.82 (the cost for a 40-pack). The price per eraser is cheaper using the two-packs.

Method 4:  
If a 40 pack costs $2.82, then half of this (20 pencils) should cost $1.41.  
But, ten 2-packs (also 20 pencils) should cost $1.20, so this is cheaper. The price per eraser is cheaper using the two-packs.

Method 5:  
Create another method you might use to determine which is the better buy.

Methods 1, 3, and 4 are correct. Answers will vary on what is most convincing. Method 2 is incorrect, because the comparison is between forty 2-packs (80 erasers) and 40 erasers. As alternative methods for Method #5, students might scale to a different value similar to methods 3 and 4, or they might set up their proportion to the rate of cost to erasers. Students
might also reason using different representations, for example graphing their solutions or setting up a table.

Connections

**Problem 2.1**

14. Find values that make each sentence correct.

   a. \( \frac{6}{14} = \frac{x}{21} = \frac{x}{28} \)
      \[ \frac{6}{14} = \frac{9}{21} = \frac{12}{28} \]

   b. \( \frac{8}{27} = \frac{8}{36} = \frac{14}{63} \)
      \[ \frac{8}{27} = \frac{8}{36} = \frac{14}{63} \]

   c. \( \frac{4}{20} = \frac{6}{30} \)
      \[ \frac{4}{20} = \frac{5}{25} = \frac{6}{30} \]

   d. \( \frac{6}{8} = \frac{15}{20} = \frac{24}{32} \)
      \[ \frac{6}{8} = \frac{15}{20} = \frac{24}{32} \]

**Problem 2.1**

15. For each diagram, write three statements comparing the areas of the shaded and unshaded regions. In one statement, use fraction ideas to express the comparison. In the second, use percent ideas. In the third, use ratio ideas.

   a. \( \frac{2}{5} \) of the square is shaded, so \( \frac{3}{5} \) of the square is unshaded. 40% of the square is shaded, so 60% is unshaded. The ratio of the shaded part to the unshaded part is 2 to 3.

   b. \( \frac{1}{9} \) of the square is shaded, so \( \frac{8}{9} \) is unshaded. Approximately 11% of the square is shaded, so 89% is unshaded. The ratio of shaded to unshaded is 1 to 8.

**Problem 2.1**

16. Multiple Choice Choose the value that makes \( \frac{18}{30} = \frac{6}{15} \) correct.

   F. 7   G. 8   H. 9   J. 10

   Answer: H

**Problem 2.1**

17. Multiple Choice Choose the value that makes \( \frac{3}{5} \) correct.

   A. 9   B. 10   C. 11   D. 12

   Answer: A

**Problem 2.2**

For exercises 13-16, rewrite each equation, replacing the variable with a number that makes a true statement.

18. \( \frac{4}{9} \times n = 1 \frac{1}{3} \)
19. \( n \times 2.25 = 90 \)
\[
40 \times 2.25 = 90
\]

20. \( n \div 15 = 120 \)
\[
1,800 \div 15 = 120
\]

21. \( 180 \div n = 15 \)
\[
180 \div 12 = 15
\]

22. Write two fractions with a product between 10 and 11.
Possible Answer: \( \frac{5}{2} \times \frac{36}{5} = 10.5 \)

23. Write two decimals with a product between 1 and 2.
Possible answers: \( 2.1 \times 0.9 = 1.89; \) or \( 5.5 \times 0.25 = 1.375 \)

Problem 2.3
The table shows the mean times that students in one seventh-grade class spend on several activities during a weekend. The data are also displayed in the stacked bar graph below the table. Use both the table and the graph for Exercises 24 and 25.

<table>
<thead>
<tr>
<th>Weekend Activities (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Category</td>
</tr>
<tr>
<td>----------------------------</td>
</tr>
<tr>
<td>Sleeping</td>
</tr>
<tr>
<td>Eating</td>
</tr>
<tr>
<td>Recreation</td>
</tr>
<tr>
<td>Talking on the Phone</td>
</tr>
<tr>
<td>Watching TV</td>
</tr>
<tr>
<td>Doing Chores and Homework</td>
</tr>
<tr>
<td>Other</td>
</tr>
</tbody>
</table>

24. The stacked bar graph was made using the data from the table. Explain how it was constructed.
Percents were calculated for boys, girls, and all students in each category. Then the percents were stacked on top of each other in the same order to show the whole 100%.
25. Suppose you are writing a report summarizing the class’s data. You have space for either the table or the graph, but not both. What is one advantage of including the table? What is one advantage of including the stacked bar graph?

The table makes it easy to compare exact hours spent on each activity. The bar graph is a quick, visual way of comparing the percentage of time spent in each category by each group. Also, comparing the heights of corresponding bands is a quick way to compare the percentage of time spent in each category between the different groups.

Problem 2.3
26. The sketches show floor plans for dorm rooms for two students and for one student.

![Floor plans]

a. Are the floor plans similar rectangles? If so, what is the scale factor? If not, why not?

Yes. The scale factor between the large room and small room is 0.75. (The ratio is 4 : 3.)

b. What is the ratio of floor areas of the two rooms (including space under the beds and desks)?

192 : 108, or 16 : 9

c. Which type of room gives more space per student?

The room for one student, as it gives 108 square feet per person while the other room gives 192 ÷ 2 = 96 square feet per person.

Extensions

Problem 2.1
27. Mammals vary in the length of their pregnancies, or gestations. *Gestation* is the time from conception to birth. Use the table to answer the questions that follow.

<table>
<thead>
<tr>
<th>Animal</th>
<th>Gestation (days)</th>
<th>Life Span (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chipmunk</td>
<td>31</td>
<td>6</td>
</tr>
<tr>
<td>Cat</td>
<td>63</td>
<td>12</td>
</tr>
<tr>
<td>Fox</td>
<td>52</td>
<td>7</td>
</tr>
<tr>
<td>Lion</td>
<td>100</td>
<td>15</td>
</tr>
<tr>
<td>Black Bear</td>
<td>219</td>
<td>18</td>
</tr>
<tr>
<td>Gorilla</td>
<td>258</td>
<td>20</td>
</tr>
<tr>
<td>Moose</td>
<td>240</td>
<td>12</td>
</tr>
<tr>
<td>Giraffe</td>
<td>425</td>
<td>10</td>
</tr>
<tr>
<td>Elephant (African)</td>
<td>660</td>
<td>35</td>
</tr>
</tbody>
</table>

*Note: The life span does not have to be converted to days to make a comparison.*

**Figure 3**

<table>
<thead>
<tr>
<th>Animal</th>
<th>Gestation (days)</th>
<th>Life Span (years)</th>
<th>Life Span (days)</th>
<th>Ratio of Life Span to Gestation (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chipmunk</td>
<td>31</td>
<td>6</td>
<td>2,190</td>
<td>2,190 : 31, or 70.6</td>
</tr>
<tr>
<td>Cat</td>
<td>63</td>
<td>12</td>
<td>4,380</td>
<td>4,380 : 63, or 69.5</td>
</tr>
<tr>
<td>Fox</td>
<td>52</td>
<td>7</td>
<td>2,555</td>
<td>2,555 : 52, or 49.1</td>
</tr>
<tr>
<td>Lion</td>
<td>100</td>
<td>15</td>
<td>5,475</td>
<td>5,475 : 100, or 54.75</td>
</tr>
<tr>
<td>Black Bear</td>
<td>219</td>
<td>18</td>
<td>6,570</td>
<td>6,570 : 219, or 30</td>
</tr>
<tr>
<td>Gorilla</td>
<td>258</td>
<td>20</td>
<td>7,300</td>
<td>7,300 : 258, or 28.3</td>
</tr>
<tr>
<td>Moose</td>
<td>240</td>
<td>12</td>
<td>4,380</td>
<td>4,380 : 240, or 18.25</td>
</tr>
<tr>
<td>Giraffe</td>
<td>425</td>
<td>10</td>
<td>3,650</td>
<td>3,650 : 425, or 8.6</td>
</tr>
<tr>
<td>Elephant</td>
<td>660</td>
<td>35</td>
<td>12,775</td>
<td>12,775 : 660, or 19.4</td>
</tr>
</tbody>
</table>

**a.** Plan a way to compare life span and gestation time for animals and use it with the data.

Ratios are a possible method of comparison.

One way to do this would be to first change life span, which is measured by years, to be measured by days. This can be done by multiplying the number of years for life span by 365 (days). Then, change the ratios into decimals in order to compare (Figure 3).

Note: The life span does not have to be converted to days to make a comparison.

**b.** Which animal has the greatest ratio of life span to gestation time? Which has the least ratio?

The greatest life span to gestation time ratio is the chipmunk, which has a ratio of 2,190 to 31, or 70.6. The least life span to gestation time ratio is the giraffe, which has a ratio of 3,650 : 425, or 8.6.

**c.** Plot the data on a coordinate graph using *(gestation, life span)* as data points. Describe any interesting patterns that you see. Decide whether there is any relation between the two variables. Explain how you reached your conclusion.

Most of the coordinates follow the pattern that as gestation increases, life span increases. This is true except for two of the mammals, the moose and giraffe. From the pattern, there does appear to be a relationship between the gestation and the life span.
d. What pattern would you expect to see in a graph if each statement were true?

i. Longer gestation time implies longer life span.
   A positive slope, going up from the left to the right, to illustrate that as \( x \) (gestation) goes up/increases, \( y \) (life span) goes up/increases.

ii. Longer gestation time implies shorter life span.
   A negative slope, going down from left to right, so as \( x \), or gestation, goes up/increases, \( y \) (life span) goes down/decreases.

Problem 2.2
28. Chemistry students analyzed the contents of rust. They found that it is made up of iron and oxygen. Tests on samples of rust gave these data.

<table>
<thead>
<tr>
<th>Amount of Rust (g)</th>
<th>Amount of Iron (g)</th>
<th>Amount of Oxygen (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>35.0</td>
<td>15.0</td>
</tr>
<tr>
<td>100</td>
<td>70.0</td>
<td>30.0</td>
</tr>
<tr>
<td>135</td>
<td>94.5</td>
<td>40.5</td>
</tr>
<tr>
<td>150</td>
<td>105.0</td>
<td>45.0</td>
</tr>
</tbody>
</table>

a. Suppose the students analyze 400 grams of rust. How much iron and how much oxygen should they find?
   280 g of iron and 120 g of oxygen. The fraction of oxygen to rust is 0.3. The fraction of iron to rust is 0.7.

b. Is the ratio of iron to oxygen the same in each sample? If so, what is it? If not, explain.
   Yes, 7:3

c. Is the ratio of iron to total rust the same in each sample? If so, what is it? If not, explain.
   Yes, 7:10
Investigation 2
Problem 2.3
29. A cider mill owner has pressed 240 liters of apple juice. He has many sizes of containers in which to pack the juice.

a. The owner wants to package all the juice in containers of the same size. Copy and complete this table to show the number of containers of each size needed to hold the juice.

<table>
<thead>
<tr>
<th>Volume of Container (liters)</th>
<th>10</th>
<th>4</th>
<th>2</th>
<th>1</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{4}$</th>
<th>$\frac{1}{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Containers Needed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answer:

<table>
<thead>
<tr>
<th>Volume of Containers (liters)</th>
<th>10</th>
<th>4</th>
<th>2</th>
<th>1</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{4}$</th>
<th>$\frac{1}{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Containers Needed</td>
<td>24</td>
<td>60</td>
<td>120</td>
<td>240</td>
<td>480</td>
<td>960</td>
<td>2,400</td>
</tr>
</tbody>
</table>

b. Write an equation that relates the volume $v$ of a container and the number $n$ of containers needed to hold 240 liters of juice.

Number needed = $240 \div Volume$, or $n = \frac{240}{v}$.

Mathematical Reflections

1. a. How do you find a unit rate or constant of proportionality in a table, a graph or an equation? A unit rate is found in the table by scaling one of the variables to “1.” In a graph the unit rate or constant of proportionality is the point (1, y) In an equation relating two variables, x and y, of the format $y = mx$, the “m” in the equation is the unit rate or constant of proportionality.

b. When are tables, graphs and equations useful? Tables and graphs are useful because they show the solutions to lots of related proportions. Once you know the unit rate, m, the equations $y = mx$ lets you substitute a value for x and find y, or vice versa.

2. How are unit rates useful? Once you know the unit rates you can use that as a multiplier to scale up a table. (Or solve an equation, as above.)

3. How is finding a unit rate like solving a proportion? When we scale a rate table like that below, we typically say we need to divide the 60 in the table, by 60, to scale this down to the 1 in the table. We then scale the 40 in the same way. So, the unit rate is $40 \div 60$ or 40/60.

<table>
<thead>
<tr>
<th>N</th>
<th>40</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>60</td>
<td>1</td>
</tr>
</tbody>
</table>
If we set up the same information in a proportion we have $40/60 = N/1$. So $N = 40/60$. In both cases we divide a value for $N$ by the corresponding value for $C$, $N/C = \text{unit rate.}$
Investigation 3
Markups, Markdowns and Measures: Using Ratios, Percents, Proportions

Mathematical and Problem-Solving Goals
Ratios, Rates and Percents: Understand ratios, rates and percents.

- Use ratios, rates, fractions, differences, and percents to write statements in a given situation comparing two quantities
- Distinguish between and use both part-part and part-whole ratios in comparisons
- Use percents to express ratios and proportions
- Recognize that rate is a special ratio that compares two measurements with different units.
- Analyze comparison statements made about quantitative data for correctness and quality
- Make judgments about which statements are most informative or best reflect a particular point of view (for example, a percent and a fraction comparison or a difference and a ratio)

Proportionality: Represent and Recognize Proportionality in Tables, Graphs and Equations

- Recognize that constant growth in a table or in a graph is related to proportional situations
- Write an equation to represent the pattern in a table of proportionally related variables
- Connect a unit rate and constant of proportionality to the equation describing a proportional situation
- Connect unit rate and constant of proportionality to a graph representing a proportional situation.

Reasoning Proportionally: Strategies for Solving Problems

- Recognize situations in which proportional reasoning is appropriate to solve the problem
- Scale a ratio, rate, percent, or fraction to make an equivalent ratio or rate (including unit rate), or to make a comparison.
- Strategically find equivalent rates (including unit rates) and ratios to solve problems
- Apply various strategies to solve for the unknown part when one part of two equal ratios is unknown, including scaling, rate tables, unit rates, equivalent ratios.
- Set up and solve proportions that arise in applications for example, finding percents in the
Mathematical Goals and Mathematical Reflections

<table>
<thead>
<tr>
<th>Goals</th>
<th>Mathematical Reflections</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apply various strategies to solve for the unknown part when one part of two equal ratios is unknown, including scaling, rate tables, unit rates, equivalent ratios.</td>
<td>1. What are some strategies you have developed for solving proportions?</td>
</tr>
<tr>
<td>Set up and solve proportions that arise in applications for example, finding percents in the context of discounts and markups, converting measurement units.</td>
<td>2. Describe a strategy for converting a rate measured in one pair of units to a rate measured in another pair of units, for example, ounces per cup to pounds per gallon, or calories per pound to calories per kilogram.</td>
</tr>
<tr>
<td>Recognize situations in which proportional reasoning is appropriate to solve the problem</td>
<td>3. How are the ideas about scaling that you used in Stretching and Shrinking the same as or different from the ideas about proportions and rates you used in Comparing and Scaling?</td>
</tr>
<tr>
<td>Connect unit rate and constant of proportionality to a graph representing a proportional situation.</td>
<td>4. What are some connections you have found among unit rates, proportions and rate tables?</td>
</tr>
</tbody>
</table>

Teaching Notes for Each Problem

3.1 Commissions, markups and discounts: Proportions with Percents

Introduction

Description of Problem 3.1

Some vocabulary is introduced in Problem 3.1 in the context of buying and selling cars. Markups, discounts, commission and tax all provide opportunities for students to work with percentages, using proportional reasoning to solve problems. Rate tables, percent bars and proportions are tools students might use; these are essentially the same strategy, scaling, but students will have preferences. Several questions ask students to find 100% of a price from given information about some other percentage of the same price. Alternative ways of finding 110% of a price are hinted at, reminding students of the Distributive Property. Students explore the result of taking two percentages in succession, to determine the net effect. Part C confronts the common misunderstanding of the result of the combination of a markup and a discount, where the same percentage is used, but the initial amounts are different.

*Focus Question:*

Investigation 3
How can we use proportions or percent tables to find 100%(or 110% etc) of a value if we know 5% (or 10% etc) of the same value?

Grouping

Problem 3.1 has several challenges: new context, new vocabulary and connections among several ideas. Groups of 2 or 3 would allow conversation and clarification of ideas.

Launch

Connecting to Prior Knowledge

Students have worked with percent bars in grade 6. Some questions about the percent bar will help recall this prior knowledge, and connect this with rate tables and proportions.

- What does the 100% mark on the percent bar represent? (The unknown ticket price.)
- Does the 8% mark seem to be in about the correct place? (Some students will be puzzled because we typically add the sales tax to the ticket price, making 108% in this case. They may need to see a model showing 100%, 108% and 8%.)
- Why is $1 written above 8% on the percent bar?
- If you know that 8% of an unknown price is $1, what other percentages of the unknown price can you figure? (We know 16% of the unknown price would be $2, 24% of the unknown price would be $3 etc. Doubling, tripling etc seems natural. Dividing by 8 to get 1% of the unknown price is perhaps less obvious.)
- Is finding out 1% of the unknown price a useful strategy?
- How is the percent table like a rate table? Like a proportion?
- How can we find the missing value in the table, or on the percent bar? Can we use a proportion? (From Inv. 2 students should have realized that the equivalent ratios in rate tables can be set equal and turned into proportions. So, from the percent table we have $1-8. &= 100$)

Issuing the Challenge

How can we use the information about percent markups and percent discounts, and our strategies for working with proportional relationships, to make informed decisions about which workplace is more advantageous for the employee or the buyer?

Explore

As you move around the room, listening to students discuss the Problem, be on the lookout for opportunities to ask about connections among percents, ratios, rate tables, percent tables and proportions. Be alert to different ways to set up and solve problems. The Summary should compare these, so students see connections among percent tables, percent bars and proportions. These are some questions you might ask.

Suggested Questions:

- How are you figuring Mia’s commission? Can you do this another way? (Students may be taking 25% of Markup. Or they may set up a table or proportion. For example, with a markup of $300, \( -25. = 300-100 \), in the form of an equation; or, in the form of a table:

<table>
<thead>
<tr>
<th>Percent of markup</th>
<th>25%</th>
<th>100%</th>
<th>1%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dollars</td>
<td>x</td>
<td>300</td>
<td>?</td>
</tr>
</tbody>
</table>

Investigation 3
How is finding 1% helpful?

- How can you work backwards from knowing 10% of the buying price to find 100% of the buying price? Can you show me what this would look like in a percent table? (*Students might use a percent bar, or percent table, or proportion. But since the numbers are easy they might just multiply by 10, and not notice that they are scaling up.*)

- What comparison is Kara making in A3a? What does the 110% refer to? Could you put this information in a percent table or on a percent bar?

- What is the 5% sales tax figured on? So the customer’s $23,000 limit has to be what percent of the selling price?

- In C1 what percentage of the buying price is the selling price? Does this help you set up a table or a proportion?

- When Otto added 15% to the buying price to get the selling price and then took 15% off the selling price why did these two 15% amounts not cancel each other out? ($20,700 is 115% of the buying price, and we want to work back to 100% of the buying price. So, “15%” in this case refers to 15% of the buying price. Otto’s discount is 15% of the selling price, a different number.)

**Summarize**

The questions in 3.1 are almost all about finding one percentage of an amount, given information about some other percentage of the same amount. For example, knowing 100% of N, how can we find 25% of N, or, knowing 110% of N how can we find 100% of N. In other words they are all about proportional reasoning. Because the percent table and percent bar encourage finding 1% and then scaling up students might not see that this is really the same as solving a proportion. They might use two scaling steps: scale down to 1% and then scale up to whatever percentage is called for. So, in A3c, if 105% of the selling price is $23,000, then 1% of the selling price is $23,000/105, or $219.05, and 100% of the selling price is $23,000 - 105 × 100. This is the same answer we would get if we set up a proportion $23,000 - 105 = x$, -100, and solved by making a common denominator of 105×100. Some students might notice that we could “collapse” the ratio on the left to one number (which is 1% of the selling price or $219.05) and then multiply by 100. This is another way to solve proportions.

**Suggested Questions**

- We have a percent table solution and another solution for A2. How are they the same or different?

- We have a proportion solution and another solution for A3. How are they the same or different?

- If you take 10% of the buying price to get the markup and then 25% of the markup to get the commission, is this the same as 2.5% (or 25% of 10%) of the buying price? Do we need to know the buying price to be able to decide if 25% of 10% of the buying price is more or less than 20% of 15% of the buying price? (*As long as these are the same buying price we do not need to know this.*)

- How many ways can we set up and solve C1? (*A proportion: $20700 - 115 = x$, -100; or a percent table:*

<table>
<thead>
<tr>
<th>Dollars</th>
<th>20700</th>
<th>B</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent (of buying)</td>
<td>115</td>
<td>100</td>
<td>1</td>
</tr>
</tbody>
</table>
Check For Understanding

The price of a hotel room is figured by adding state and city taxes to the advertised price. The taxes are 30% of the advertised price, and the bill is $112. Find the advertised price by using a proportion and one other method.

$112 = 130 \times \frac{1}{100}$, $100 \times 130$. So $P = \frac{112 \times 100 - 130}{100}$. Common denominator method: $112 \times 100 - 130 = 100$. So $P = \frac{112 \times 100}{130}$.

Table method:

<table>
<thead>
<tr>
<th>Dollars</th>
<th>112</th>
<th>$P$</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of advertised price</td>
<td>130</td>
<td>100</td>
<td>1</td>
</tr>
</tbody>
</table>

Focus Question:

How can we us proportions or percent tables to find 100% (or 110% etc) of a value if we know 5% (or 10% etc) of the same value?

Answers to Problem 3.1

A. 1. Commission as in table below.

<table>
<thead>
<tr>
<th>Make</th>
<th>Markup</th>
<th>Commission</th>
<th>Buying price</th>
<th>Selling price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ford</td>
<td>$300</td>
<td>$75</td>
<td>$3000</td>
<td>$3300</td>
</tr>
<tr>
<td>Toyota</td>
<td>$500</td>
<td>$125</td>
<td>$5000</td>
<td>$5500</td>
</tr>
<tr>
<td>Pontiac</td>
<td>$950</td>
<td>$237.50</td>
<td>$9500</td>
<td>$10, 450</td>
</tr>
</tbody>
</table>

2.a. Buying prices as in table. Students might set up a proportion markup/10% = buying price/100%.

2.b. Selling prices as in table. Students are likely to add markup to buying price to get selling price. But they could set up a proportion markup/10% = selling price/110%.

3. a. Kara is correct. The markup is 10% of the buying price. The selling price = markup + buying price = 10% of buying price + 100% of buying price = 110% buying price. So the ratio selling price: 110% = buying price: 100% is correct.

3. b. Mia is also correct. But when she finds M from this proportion she still has to add this to the buying price to get the selling price.

3. c. The customer’s spending limit is 105% of the maximum selling price, that is, $23000 = 105% of selling price. As a proportion,

$23000/105 = S/100$. $S \approx 21904$. 

Investigation 3
B. 1. Kara will give Mia 25% of the markup, or 25% of (10% of $20500) = $512.50. Students might calculate the markup first and then take 25% of that; or they may say 25% of 10% = 2.5%, and then calculate 2.5% of 20500 = $512.50.

Otto will give Mia 20% of the markup, or 20% of (15% of $20500) = $615.

Students might not calculate the commission; they may just compare 25% of 10%, which is 2.5%, and 20% of 15%, which is 3%.

2. Students might make a difference statement; Otto will pay Mia $102.50 more than Kara. Or they may use ratios; 3%:2.5% = 6:5 = 1.2:1, so Otto will pay Mia $6 for every $5 Kara pays, or Otto will pay 120% of what Kara will pay.

C. 1. $18,000. $20700: 115% = B: 100%. Students might scale this to 1%, so $180:1% = B: 100%. If 1% of the buying price is $180, the buying price must be $18000.

2. 15% of 20700 = 3105 If Otto discounts his cars by 15% then the Mercedes will have a selling price of $20,700 - $3105 = $17595. This is below the buying price so there is no markup.

Note: this addresses a common misconception. If you raise an original value by 15% and then take 15% off the new value you will not be back where you started. Raising a value of X by 15% gives 115% of X. Taking 15% off this new value gives 85%(115% of X). 85% x 115% is not the same as 100%; it is less than 100%.

3.2 Measuring to the Unit: Measurement Conversions

Introduction

Description of Problem 3.2

In Problem 3.2 Students meet proportional situations where the units are mixed. They have to decide on a unit of measure (hours or minutes, but not both, for example) and then set up a proportion. Sometimes they have to convert units from different measurement systems. This provides opportunities to use rate tables, proportions, or unit rates, and to see connections. Part C makes explicit another strategy for solving proportions: condensing one ratio in the proportion to one number, or to a unit rate.

Focus Question:

How are rate tables, proportions and unit rates connected?

Grouping

Part A can be used as a check on how individual students are setting up proportions and rate tables. Students could work individually on this part. Parts B and C require discussion, so groups of 2 – 4 would be advisable.

Launch part A

Connecting to Prior Knowledge

Questions to ask:

- What question does the proportion \( \frac{1}{-2.5} = \frac{-48}{-48} \) ask?
• How can we solve this proportion? (*There is no easy scale factor, so a common denominator strategy could be used.* \( \frac{1}{2} \times 48 = 2.5 \times 48 \Rightarrow \frac{1}{2} \times 48 \times 2.5 \). Since denominators are now the same, \( 48 = 2.5x \), so \( x = \frac{48}{2.5} = 19.2 \). Or a rate table would show:

<table>
<thead>
<tr>
<th>inches</th>
<th>1</th>
<th>1÷2.5 = 0.4</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>centimeters</td>
<td>2.5</td>
<td>1</td>
<td>48</td>
</tr>
</tbody>
</table>

• What does the answer for \( x \), in the above proportion or rate table, tell you?

**Issuing the Challenge for Part A**

You have worked with several strategies for solving problems that deal with proportional situations. Try these out on Part A by yourself.

**Explore part A**

As you move around the room, observing students work on part A, these are some questions you might ask.

**Suggested Questions:**

• What are the units in your table (or proportion)?
• Will scaling up work for your proportion or will you have to use a common denominator strategy?
• Do you need to find a unit rate? What scale factor will give you a unit rate? Is there another way to find a unit rate? What does the unit rate tell you?

**Summarize part A**

There is no need to do a full summary. You will know from your observations which students still need help in executing a strategy successfully. Having students post and compare correct solutions should suffice.

**Launch parts B and C – use as needed.**

Part B pushes students to realize that dividing numerator by denominator of a rate will give you a unit rate, in all cases. They have seen this before in Problem 2.3, but the fractions in this question may throw them off. It is easy to see division is needed when whole numbers are involved: 16 miles in 4 hours, how many miles in 1 hour? Or 286 calories in 5 ounces, how many calories in 1 ounce? But in the Sean’s walking rate of \( \frac{3}{4} \) mile in \( \frac{1}{4} \) hour, the natural thing is to scale this up by multiplying by \( \frac{4}{4} \), to give 3 miles in 1 hour. However, examining a rate table shows that dividing will work in all cases. You might have to remind your students about this. If you think that your students do not need this reminder you can skip this mini-launch.

**Suggested Questions**

• When we explored unit rates in Investigation 2 we solved problems like *What is the unit rate per box of pasta if 6 boxes cost $5? “What is the unit rate per dollar”*

| Dollars | 5 | 1 | D |

Investigation 3
What are the solutions for B and D? (Students might divide correctly, but if they are unsure about this, then setting up a proportion will clarify. B = 6/5 and D = 5/6.) Both unit rates are made by dividing. Is this always possible?

- When you look at Sean’s problem about walking rates in a rate table how would you make a unit rate?

| miles | 3/4 | ? |
| hours | 1/4 | 1 |

(Students will probably say scale up ¼ hour by factor of 4 to get 1 hour) Is there another operation that is equivalent to multiplying by a factor of 4?

**Grouping**

2-4 students.

**Issuing the Challenge**

You are going to explore another connection between rates and proportions.

**Explore parts B and C**

**Questions to ask**

- What does the division in B1 tell you? What would a division in B2 tell you? (12 beads/5 inches = 2.4 beads per inch. Or 5 inches/12 beads = 0.416 inches per bead.)

- Does a unit rate help you figure out the number of beads in 1 foot? Which unit rate helps?

- What do all the parts of the proportion in C1 mean?

- It looks easier to work with the proportion ,3-3,-1,-1,-3-. How might you solve this? (Students might use fact families: x ÷ 1 1/3 = 3, so x = 3 × 1 1/3 = 4. Or they might see that scaling up with ,1,1,-3,-1,-3.. will solve the proportion.)

- How does completing the rate table in C2 answer Sean’s problem?

**Summary**

There are two mathematical points to bring out in the summary: that dividing the parts of a rate will give a unit rate, every time; and that this is one way to rewrite a proportion. Once the proportion has been written as A = , then students can use fact family ideas. (A and B are factor pairs and, multiplied, make the product X.) This is exactly what happens when a rate table is used to scale to a unit rate, and then scale again to find a missing part of a ratio.

**Questions to ask**

- How is completing the rate table in C2 like solving the proportion in C1?

- Can this “unit rate” strategy be used on all proportions? What about A5 and A4?

 How would these look as proportions? As rate tables?

(A5: ,27-6,-16. or
### Calories

<table>
<thead>
<tr>
<th>Calories</th>
<th>276</th>
<th>?</th>
<th>C</th>
</tr>
</thead>
</table>

### Ounces

| Ounces | 6 | 1 | 16 |

---

\[ A4: \frac{3}{2} \cdot 3 \cdot \text{=} \quad /1 \]

<table>
<thead>
<tr>
<th>Acres</th>
<th>3</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tanks</td>
<td>2/3</td>
<td>1</td>
</tr>
</tbody>
</table>

---

**Check for Understanding**

What proportions does this rate table help to solve?

<table>
<thead>
<tr>
<th>Miles walked</th>
<th>5</th>
<th>?</th>
<th>M</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hours taken</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3/4</td>
</tr>
</tbody>
</table>

---

**Focus Question:**

*How are rate tables, proportions and unit rates connected?*

---

### Answers to Problem 3.2

**A.**

1. Kate walked \(3\ 1/8\) miles in \(1\ 1/4\) hour.

2. Sean walked 4 miles in \(1\ 1/3\) hours.

3. A gallon of milk has 128 grams of fat.

4. Sean can cut \(4\ 1/2\) one-acre lawns with one tank of gas.

5. One pound of chicken has 736 calories.

6. 28 beads will fit on one foot of necklace, with room for \(4/5\) of another.

7. 50 beads take up about 52 1/12 centimeters. He will need 53 centimeters of string.

**B.**

1. Sean’s rate is \(3/4\) mile per \(1/4\) hour. The expression he wrote gives a unit rate, 3 miles per 1 hour

2. Scott’s necklace has 12 beads per 5 inches. If Scott divides \(12/5\) he will get a unit rate of 2.4 beads per inch. He really wants the unit rate # beads per 1 foot. So he could multiply this by 12, to get 28.8 beads per foot. Or he could have rewritten the units in the first place as 12 beads per 5/12 of a foot, and then divided \(12 \div 5/12 = 28.8\) beads per foot.

Sean’s method always gets a unit rate. You just have to be careful you label the units you are using.

**C.**

1. Sean’s strategy is the same as setting up the proportion. When Sean divides the two parts of the ratio he knows (3/4 mile; \(1/4\) hour) he gets a unit rate. Doing the same kind of division in a
proportion, for the ratio where you know two parts, will likewise give a unit rate as one side of the equation. This may make an easier equation to solve.

2. Sean’s strategy is the same as what we do when we scale a rate in a table. In the table below we can scale up by multiplying the \(\frac{1}{4}\) by 4 to get 1, and then multiplying \(\frac{3}{4}\) by 4 to get 3. Or we can also think of this as dividing the \(\frac{1}{4}\) by \(\frac{1}{4}\) to get 1 and then dividing \(\frac{3}{4}\) by \(\frac{1}{4}\) to get 3. Once we have the unit rate we can scale this up to find the distance gone in 1 1/3 hours.

<table>
<thead>
<tr>
<th>Distance in miles</th>
<th>(\frac{3}{4})</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Times in hours</td>
<td>(\frac{1}{4})</td>
<td>1</td>
</tr>
</tbody>
</table>

3.3 Mixing it Up: Connecting Ratios, Rates, Percents and Proportions

**Introduction**

**Description of Problem 3.3**

Problem 3.3 presents information about a mixture, echoing the Orange Juice problem, in which the ratio of ingredients is given in percents. The table looks a little different because both part-to-part and part-to-whole ratios are shown. Students use rate tables, unit rates, equations, graphs, and/or proportions to answer questions about this mixture. Sometimes they reason about part-to-part ratios and sometimes about part-to-whole ratios. The power of a graph is that it shows many solutions for an equation. Students will do more with this idea in the next unit, Moving Straight Ahead. An alternative way to solve proportions is offered in part C, similar to the rate table strategy introduced in 3.2.

**Focus Question:**

*If a mixture is 60% fiber and 40% protein and has 120 cups fiber in the total mix, would you use scaling or rate tables or proportions or equations or graphs to find out how many total cups are in the mixture?*

**Grouping**

2 – 4 students.

**Launch**

**Connecting to Prior Knowledge**

Questions to ask:

- When we explored the orange juice problem, one recipe was 2 cans concentrate to 3 cans water. What percentage of the mixture is concentrate? What percentage is water?
- What would a rate table look like for this recipe? What would a unit rate mean?

\[(1.5 \text{ cans water per 1 can concentrate. Or } 2/3 \text{ can concentrate per 1 cup water.}\]

<table>
<thead>
<tr>
<th>Cans Concentrate</th>
<th>2</th>
<th>4 etc</th>
<th>1</th>
<th>2/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cans Water</td>
<td>3</td>
<td>6 etc</td>
<td>1.5</td>
<td>1</td>
</tr>
</tbody>
</table>
• Do the percentages of concentrate and water appear in your rate table? *(We could scale up 2:3 to 40:60.)*

• What would solving this proportion tell you about mixing orange juice? *2:3=4:1-2:.- ? How would you solve it?* *(How many cans of water do we need if there are 4 ½ cans of concentrate? Rewrite as 3-2-= 4-1-2. and then make common denominators, 27-18=4 -18, so 4N = 27, so N = 6 ¾. Or work from a unit rate: For 1 can concentrate we need 1.5 cans water, so for 4 ½ cans concentrate we need 4 ½ × 1.5 cans water.)*

• What would solving this proportion tell you about mixing orange juice? *40-60.= -5. ? (How many cans of concentrate do we need if there are 5 cans water? The 40:60 ratio is the same as 2:3)*

• Are the ratios in these proportions part-to-part or part-to-whole? *(Part to part)*

*Issuing the Challenge*

You know a lot about rate tables, proportions, equations and graphs. You are going to use these to solve problems about mixing food for chimpanzees, not juice for campers. If you don’t get the mix right the chimps will not be healthy!

*Explore*

As you move around the room, listening to students discuss the Problem, these are some questions you might ask.

*Suggested Questions:*

• In A2, do the unit rates help you write the equations relating F and P? How can you check you have the variables in the right places?

• In A3a, does your ratio represent part-to-part or part-to-whole?

• In A3b, did you try setting up a proportion? Did you try using the rate table?

• In B, Ming wants to keep the 30 scoops of protein the same. She needs to figure out how many scoops of fiber should go with that amount of protein. Does this help you set up a proportion? Are you thinking of part-to-part or part-to-whole ratios?

• What ratios are related to the two points marked on the graph in B2? Can you ask questions that each of the points would answer?

• Suppose the equation in C2 was simpler, say , -2.=8. What other sentences are in the same fact family as this sentence? 2 and 8 are a factor pair for x. Does this help you think about the sentence Ming wrote?

*Summarize*

There are three somewhat new things in this Problem: the ratio is given in percentages, the table shows both part-to-part ratios and part-to-whole ratios, and the solution for the proportion in B1 is found on a graph in B2. Ming’s method of solving, in C, where she makes one side of the proportion into a single number by dividing, is similar to something students saw in 3.2. This Problem gives students a last opportunity to see the power of the rate table (and unit rates) in finding unknown parts or ratios (aka solving proportions). The summary should compare methods of solving proportions, and connect this to the graph.

*Suggested Questions*
• Someone wrote \( F = \frac{2}{3} P \) and \( P = \frac{2}{3} F \) and \( F = 1.5P \) and \( P = 1.5F \) for the equations in A2. How do you know which are correct? (We could substitute known pairs of \((P, F)\) to check. Or we can say \(2/3\) is a unit rate: \(2/3\) cup Fiber per 1 cup Protein. So for every known value of \( P \) we multiply by \(2/3\) to find \( F \). Likewise \(1.5\) is a unit rate: \(1.5\) cups Protein per 1 cup Fiber. So for every known value of \( F \) we multiply by \(1.5\) to find \( P \).)

• Compare a table way of solving A3b with setting up and solving a proportion. (We need the ratio of high fiber to total mix. We know this is 40:100. So the proportion is \( \frac{40}{100} = -\frac{125}{125} \). Now we need to scale or make common denominators. Using a table:

<table>
<thead>
<tr>
<th>High Fiber</th>
<th>40</th>
<th>( \frac{2}{3} )</th>
<th>( \frac{2}{5} )</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Mix</td>
<td>100</td>
<td>( \frac{1}{2/3} )</td>
<td>1</td>
<td>125</td>
</tr>
</tbody>
</table>

Using the unit rates we found in A (shaded column) don’t help much. We need another unit rate to show \( \frac{2}{5} \) scoops of Fiber per 1 cup total mix. Or we could scale 100 to become 125 without making a unit rate.)

• What other points are on the graph in B2? Can you ask a question for each point, that the coordinates would answer?

  • A3 is about finding how many scoops of fiber are needed for 48 scoops of protein in the baby chimp mix. The answer was 32 scoops of fiber. But that pair is not on the graph in B. Why? (The pairs on the graph in B give answers about the adult chimp mix. The equations for the baby chimp mix are \( F = \frac{2}{3} P \) or \( P = 1.5F \). The equation for the adult chimp mix is \( P = \frac{2}{3} F \). The graph matches \( P = \frac{2}{3} F \).)

Check For Understanding

Suppose Ming has 35 scoops of Protein. She wants to make a mix that is 40% Fiber and 60% protein. How many cups of Fiber should she add? Solve this as many ways as you can.

Focus Question:

If a mixture is 60% fiber and 40% protein and has 120 cups fiber in the total mix, would you use scaling or rate tables or proportions or equations or graphs to find out how many total cups are in the mixture?

Answers to Problem 3.3

A. 1.

<table>
<thead>
<tr>
<th>Scoops of High Fiber Food</th>
<th>40</th>
<th>40/60 or 2/3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scoops of High Protein Food</td>
<td>60</td>
<td>1</td>
<td>60/40 or 3/2</td>
</tr>
<tr>
<td>Total Scoops in Mix</td>
<td>100</td>
<td>1 ( \frac{2}{3} )</td>
<td>2 ( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

2. \( F = \frac{2}{3} P \) or \( P = 1.5F \).

3. a. 32 cups. Students could solve the proportion, 48: \( F = 60:40 \), or use the equation, \( F = \frac{2}{3} \times 48 \). Or use a rate table.
b. 50 cups. Students could solve the proportion $F: 125 = 40: 100$. (This is a part–whole ratio.) Or they might say that the total number of scoops is $2\frac{1}{2}$ times the number of scoops of Fiber. This would be the equation $125 = 2.5F$. Or they might use a rate table.

c. See above.

B. 1. Students might solve the proportion $F: 30 = 60:40$. This says we want the scoops of Fiber to be 60% of the mix, and we want the 30 scoops of Protein already added to be 40% of the mix. (These are part-to-part ratios) We could scale this to $4F: 120 = 180: 120$, so $4F = 180$, so we need to have 45 scoops of fiber in the mix. So we need another 25 scoops of fiber.

2. 
   - Students might make a rate table for the adult chimp mix and match the pairs to the graph. Or they might say that $2/3$ is a unit rate, meaning $2/3$ cup Protein for 1 cup Fiber. This matches the ratio 40 cups protein for 60 cup Fiber. So this graph shows pairs that fit this proportional relationship.
   - Ming needs to find out how many scoops of Fiber she should have in a mix that has 30 cups of Protein. She needs to find a point on the graph that is $(F, P) = (F, 30)$. This should give her $F = 45$ cups.

C. 1. She is correct. Dividing 60 by 40 gives us 1.5, meaning 1.5 scoops of fiber per 1 scoop of protein.

2. How might Ming finish solving $\frac{x}{30} = 1.5$? She might think “$x$ must be 1.5 times 30.” Or she might scale “$x : 30 = 3 : 2 = 45: 30$.”


4. Students might complete Ming’s method, by thinking “$x$ must be 3.1 times 4.24.” Or they might use a common denominator method:

\[
\frac{2.2x}{4.24 \times 2.2} = \frac{6.82 \times 4.24}{2.2 \times 4.24}
\]

Since the denominators are equal, $2.2x = 6.82 \times 4.24$, so $x = 13.144$. Students might also scale everything up to whole numbers before embarking on making common denominators.

### Applications, Connections and Extensions

#### Applications

**Problem 3.1**

1. Determine the sales tax for each situation.
   
   a) a new shirt for $21.00 at 5% sales tax.

   \[.05 \times 21.00 = 1.05\]

**Investigation 3**
b) a new bicycle for $45.90 at 7% sales tax.
\[0.07 \times 45.90 = 3.21\] (rounded value)
c) a new pair of shoes for $67.50 at 6% sales tax
\[0.06 \times 67.50 = 4.05\]
d) a new laptop for $299.99 at 8% sales tax
\[0.08 \times 299.99 = 24.00\] (rounded value – note the sales price is so close to $300.00 that the tax value is the same as if it was $300.00)
e) a video game for $39.95 at 4% sales tax
\[0.04 \times 39.95 = 1.60\] (rounded value – note the sales price is so close to $40.00 that the tax value is the same as if it was $40.00)

**Problem 3.1**

2. Bennett was trying to solve problem part (a) of Exercise 1 in many different ways. Which of the following strategies are appropriate ways to solve part (a)? Of the one(s) that are correct, which one makes the most sense to you? Explain.

   i) 5% sales tax means that for every dollar, you spend a nickel in tax. For 21 dollars, then you are really paying 21 nickels worth of tax.

   ii) Set up a proportion and solve for the missing value:

   \[
   \frac{0.05}{1.00} = \frac{x}{21.00}
   \]

   iii) I know that 10% of $21.00 is $2.10, so 5% would be half of $2.10.

   iv) 5% is the same as \(1/20\). To find the amount of tax for $21.00 divide it by 20.

   v) 1% of $21.00 is $0.21, so 5% of $21.00 would be 5 x $0.21.

   All 5 strategies are correct. Students may choose different strategies for which one makes the most sense to them. For example, (iii) and (iv) are pretty straightforward, however they do not generalize as easily as (i) and (ii) and (v).

**Problem 3.1**

3. Your group celebrates birthdays by going out to a restaurant for pizza. Everybody chips in to the fund before you order.

   a. Your group has $63 altogether for pizza. The tax is 5%. What is the maximum amount your group can spend and not go over $63?

   If we let $$S$$ represent the spending limit before tax, then $63 has to be no more than 105% of this limit. As a proportion,
63/105 = S/100, so S = 60. If the bill without tax is $60 then the group can cover this with $63.

b. You want to leave a 15% tip on the price of the food before sales tax. What is the maximum amount your group can spend, including tax and tip, and not go over $63? Explain your reasoning.

If we let $S$ represent the spending limit before tax and tip, then $63$ has to be no more than 120% of this limit. As a proportion,

\[
63/120 = S/100, \text{ so } S = 52.5. \text{ If the bill without tax or tip is } 52.50 \text{ then the group can cover this with } 63.
\]

**Problem 3.1**

4. Ernesto is trying to estimate sales tax on each of the situations. Which estimates seem the most reasonable?

   a) 5% tax on a $42.00 purchase
      i) under $2.00  
      ii) exactly $2.00  
      iii) over $2.00

      iii is the correct answer, 5% of $40.00 is $2.00, so 5% of $42.00 would be over $2.00.

   b) 9% tax on a $59.99 purchase
      i) under $6.00  
      ii) exactly $6.00  
      iii) over $6.00

      i is the correct answer, 10% of $60.00 is $6.00, so 9% of $59.99 would be less than $6.00.

   c) 5.5% tax on a $309.95 purchase
      i) under $15.00  
      ii) exactly $15.00  
      iii) over $15.00

      iii is the correct answer, 5% of $300.00 is $15.00, so 5.5% of $309.95 would be over $15.00.

**Problem 3.1 (New) (per Betty, #5 - 8 could be used as a partner quiz)**

5. In exercises 4 - 7, Bill’s Bikes specializes in new and used bikes. When Bill buys a bike from someone, he fixes and cleans it then marks up the price by 80%. The salesperson selling the bike gets a 25% commission on the markup. Determine the missing values in the table. Determine the missing values in the table.

<table>
<thead>
<tr>
<th>Purchase Cost (what Bill pays for the bike)</th>
<th>Markup Price (80% of purchase price)</th>
<th>Retail Price (what Bill sells it for)</th>
<th>Commission (25% of the markup)</th>
<th>Profit for the shop (how much they make on the sale)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100</td>
<td>$80</td>
<td>$180</td>
<td>$20</td>
<td>$60</td>
</tr>
<tr>
<td>$55</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$125</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**ANSWERS:**

Investigation 3
<table>
<thead>
<tr>
<th>for the bike)</th>
<th>price)</th>
<th>markup)</th>
<th>make on the sale)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100</td>
<td>$80</td>
<td>$20</td>
<td>$60</td>
</tr>
<tr>
<td>$10</td>
<td>$8</td>
<td>$2</td>
<td>$6</td>
</tr>
<tr>
<td>$55</td>
<td>$44</td>
<td>$11</td>
<td>$33</td>
</tr>
<tr>
<td>$125</td>
<td>$100</td>
<td>$25</td>
<td>$75</td>
</tr>
</tbody>
</table>

6. In each of the arrows write a rule of how you get from the starting value to the next value.

```
Purchase Cost
\downarrow \times 0.80
Markup
\downarrow \times 2.25
Retail Price
\downarrow \times 0.75
Profit
\downarrow \times 3

Answers:
```

Investigation 3
7.

   a) If you knew only the markup price, how would you determine the purchase cost?
      The relationship is Markup = .80 x Purchase. Solving this for the purchase price, we have
      \[ \text{Purchase} = \text{Markup} ÷ .80 = \text{Markup} \times 1.25 \]

   b) If you knew only the retail price, how would you determine the purchase cost?
      The relationship is Retail = Purchase x .80 x 2.25 = Purchase x 1.80.
      \[ \text{Purchase} = \text{Retail} ÷ 1.80 = \text{Retail} ÷ \frac{9}{5} = \text{Retail} \times \frac{5}{9} \]

   c) If you knew the commission, how would you determine the markup price?
      The relationship is Commission = .25 x Markup. Solving this for the markup price, we have
      \[ \text{Markup} = \text{Commission} ÷ .25 = \text{Commission} \times 4 \]

   d) If you knew only the profit, how would you determine the commission?
      The relationship is Profit = Markup – Commission = (Commission x 4) – Commission =
      Commission x 3. This means that Commission = Profit ÷ 3.

8. Determine the missing values using an appropriate strategy.

<table>
<thead>
<tr>
<th>Purchase Cost (what Bill pays for the bike)</th>
<th>Markup Price (80% of purchase price)</th>
<th>Retail Price (what Bill sells it for)</th>
<th>Commission (25% of the markup)</th>
<th>Profit for the shop (how much they make on the sale)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$48</td>
<td>$252</td>
<td>$14.40</td>
<td></td>
<td>$54</td>
</tr>
<tr>
<td>( n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answers:

<table>
<thead>
<tr>
<th>Purchase Cost (what Bill pays for the bike)</th>
<th>Markup Price (80% of purchase price)</th>
<th>Retail Price (what Bill sells it for)</th>
<th>Commission (25% of the markup)</th>
<th>Profit for the shop (how much they make on the sale)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$60</td>
<td>$48</td>
<td>$108</td>
<td>$12</td>
<td>$36</td>
</tr>
<tr>
<td>$140</td>
<td>$112</td>
<td>$252</td>
<td>$28</td>
<td>$84</td>
</tr>
<tr>
<td>$72</td>
<td>$57.60</td>
<td>$129.60</td>
<td>$14.40</td>
<td>$43.20</td>
</tr>
<tr>
<td>$90</td>
<td>$72</td>
<td>$162</td>
<td>$18</td>
<td>$54</td>
</tr>
<tr>
<td>( n )</td>
<td>.80( n ) = \frac{4n}{5}</td>
<td>1.80( n ) = \frac{9n}{5}</td>
<td>.20( n ) = \frac{n}{5}</td>
<td>.60( n ) = \frac{3n}{5}</td>
</tr>
</tbody>
</table>

Problem 3.2

9. Solve each of the following conversion problems.

   a) Allen ran 8 miles in 3 hours at a steady pace. How long did it take him to run 3 miles?
8 miles in 3 hours is proportional to 1 mile in 37.5 minutes. 3 miles would take 37.5 x 3 = 112.5 minutes.

b) Maren walked 3/5 mile in 24 minutes at a steady pace. How long did it take her to walk 2 miles?

3/5 mile in 20 minutes is proportional to 1/5 mile in 8 minutes. 5/5 mile = 1 mile would take 40 minutes.

c) If half an avocado has 160 calories, how many calories are in a bag with a dozen avocados?

½ an avocado has 160 calories, and 1 dozen avocados = 24 half avocados. There are 24 x 160 = 3,840 calories

d) In a certain recipe that are 1.5 grams of fat in 1 tablespoon of hummus. How many grams of fat are in 2 ½ cups of hummus (16 tablespoons = 1 cup)?

There are 2.5 x 16 = 40 tablespoons in 2.5 cups. 1.5g x 40 = 60 grams of fat.

**Problem 3.2**

10. While most of the world uses the metric system, the United States is one of the few countries to still use the English system. There are many older conversions that are not used as commonly any more, or they may be used in specific situations. For example *hands* are still used in measuring the heights of horses, and some people measure fences or barns using *rods*.

<table>
<thead>
<tr>
<th>Common Conversions</th>
<th>Other Conversions</th>
<th>Other Conversions</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 inches = 1 foot</td>
<td>1 furlong = 220 yards</td>
<td>1 rod = 5.5 yards</td>
</tr>
<tr>
<td>3 feet = 1 yard</td>
<td>1 furlong = 10 chains</td>
<td>16 nails = 1 yard</td>
</tr>
<tr>
<td>5,280 feet = 1 mile</td>
<td>1 furlong = 1000 links</td>
<td>4 palms = 1 foot</td>
</tr>
<tr>
<td>1,760 yards = 1 mile</td>
<td>1 furlong = 40 rods</td>
<td>3 hands = 1 foot</td>
</tr>
</tbody>
</table>

Use the conversions above to predict how long it will take each person to walk 1 mile.

<table>
<thead>
<tr>
<th>Distance &amp; Time</th>
<th>Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 1584 feet in 3 minutes</td>
<td>1 mile in ?</td>
</tr>
<tr>
<td>b) 2 furlongs in 10 minutes</td>
<td>1 mile in ?</td>
</tr>
<tr>
<td>c) 1500 links in 12 minutes</td>
<td>1 mile in ?</td>
</tr>
<tr>
<td>d) 4 rods in 11 seconds</td>
<td>1 mile in ?</td>
</tr>
<tr>
<td>e) 5 chains 1 minute</td>
<td>1 mile in ?</td>
</tr>
</tbody>
</table>

For all of these problems, students should set up some sort of an equation and solve, perhaps using a proportion or some other strategy.

a) At a rate of 1584 feet in 3 minutes, it would take 1 minute to go 528 feet. So 5280 feet = 1 mile would take 10 minutes.

b) There are 8 furlongs in a mile, at the rate of 1 furlong in 5 minutes it would take 40 minutes to walk 1 mile.

c) 1500 links = 1.5 furlongs. 1 furlong takes 8 minutes, so 8 furlongs would take 64 minutes.
d) 4 rods = 22 yards in 11 seconds which is proportional to 2 yards every second. Since 2 x 880 = 1,760 and 1,760 yards = 1 mile, it would take 880 seconds = 14 minutes and 40 seconds to walk 1 mile.

e) 5 chains = \( \frac{1}{2} \) furlong. Multiplying \( \frac{1}{2} \) furlong by 16 results in 8 furlongs = 1 mile. If 5 chains is walked in 1 minute then it would take 16 minutes to walk 1 mile.

**Problem 3.2**

11. For each proportion state what conversion is represented, then solve for \( x \).

a) \[
\frac{16 \text{ ounces}}{1 \text{ pound}} = \frac{x}{3 \frac{1}{2} \text{ pounds}}
\]

b) \[
\frac{1 \text{ gallon}}{16 \text{ cups}} = \frac{x}{36 \text{ cups}}
\]

c) \[
\frac{x}{12.5 \text{ cups}} = \frac{8 \text{ fluid ounces}}{1 \text{ cup}}
\]

a) \( x \) represents the number of ounces in 1 pound, \( x = 56 \) ounces.

b) \( x \) represents the number of gallons in 36 cups. \( x = 2 \frac{1}{4} \) gallons

Cc) \( x \) represents the number of fluid ounces in 12 \( \frac{1}{2} \) cups. \( x = 100 \) fluid ounces

**Problem 3.2**

12. Set-up a proportion to solve each of the following conversion problems.

a) How many ounces equal 10 \( \frac{1}{2} \) pounds

\[
\frac{16 \text{ ounces}}{1 \text{ pound}} = \frac{x}{10 \frac{1}{2} \text{ pounds}}
\]

\( x = 168 \) ounces

b) How many cups equal 55 gallons

\[
\frac{1 \text{ gallon}}{16 \text{ cups}} = \frac{55 \text{ gallons}}{x}
\]

\( x = 880 \) cups

c) About how many pounds equals 60 kilograms

\[
\frac{1 \text{ kg}}{2.2 \text{ lbs.}} \approx \frac{60 \text{ kg}}{x}
\]

\( x \approx 132 \) lbs

**Problem 3.3**

Investigation 3
13. A few students were trying to solve part C.4 in Problem 3.3. Which of these methods work correctly? Of the ones that are correct which one makes the most sense to you? Explain.

Alicia’s Method:

First, I simplified the fraction on the right.

\[ \frac{x}{4.24} = 3.1 \]

Then, I multiplied 3.1 by 4.24 to get \( x \).

Brandon’s Method:

I multiplied all the values by 100 to eliminate the decimals.

\[ \frac{100x}{424} = \frac{682}{220} \]

Then I multiplied both sides by 424.

\[ 100x = \frac{682 \times 424}{220} \]

\[ 100x = 1314.4 \]

Then I divided both sides by 100.

\[ x = \frac{1314.4}{100} \]

Charlene’s Method: I figured out that 6.82 – 2.2 = 4.62, so the numerator in the right fraction was 4.62 bigger than the denominator. This means that \( x = 4.24 + 4.62 = 8.86 \).

Both Alicia and Brandon’s methods are correct. In Alicia’s method, simplifying one side allows you to solve the problem by “undoing” the division on the left side. In Brandon’s method, this conversion works because you are simply scaling each of the values by 100, which does not change the multiplicative relationship between quantities in the proportion (note: there is nothing special about 100, any non-zero quantity will work the same way). Charlene’s method does not work because the relationship is multiplicative not additive.

**Problem 3.3**

For exercises 13-14, use what you learned in Part C.1 of Problem 3.3. Ming said that the ratio \( \frac{60\%}{40\%} \) was the same as 1.5.

14. Given the following ratios of high fiber to high protein mix. For each of the ratios given below re-write as an equivalent unit rate.
a) 75% high fiber to 25% high protein mix  
   unit rate = 3  
b) 80% high fiber to 20% high protein mix  
   unit rate = 4  
c) 85% high fiber to 15% high protein mix  
   unit rate = 5 \frac{2}{3}  
d) 95% high fiber to 5% high protein mix  
   unit rate = 19

15. Suppose you knew the unit rate related to the ratio of high fiber to high protein mix, re-write the unit rate as a ratio of high fiber to high protein mix.

a) Unit rate: 1  
   50% high fiber to 50% high protein mix  
b) Unit rate: \frac{1}{3}  
   25% high fiber to 75% high protein mix  
c) Unit rate: 9  
   90% high fiber to 10% high protein mix

Problem 3.3

16. Suppose you have 24 scoops of high fiber food.

a) How many scoops of high protein food should you mix if you were using it for baby chimps?  
   36, 24 \times 1.5 = 36.  
b) How many scoops of high protein food should you mix if you were using it for adult chimps?  
   16, 24 \times \frac{2}{3} = 16

Connections

Problem 3.1

17. After working through some markup and commission problems Claire and Pam were wondering about the following two situations:

I) marking up the price 25% then getting a 10% commission on the markup
II) marking up the price 10% then getting a 25% commission on the markup

Will these two give the same amount of payment for the commission or will one be higher? If so, which one pays a higher commission?

Both pay the same amount. Suppose the starting price is $P$. Then the markup in situation I will be $.25P$. The sales person gets 10% of the markup, so commission = $.10(.25P) = .025P$. In Situation II, the markup is $.10P$. The commission is $.25(.10P) = .025P$.

Problem 3.1

18. Erin notices something interesting while working through the tax problems in exercise 1. For example in part a, she took $.05 \times $21.00 = $1.05$. If she wanted to find the total cost she would add $21.00 + $1.05 = $22.05$. She re-writes this as $(1 \times $21.00) + (.05 \times $21.00) = 1.05 \times $21.00$ using the distributive property. How might you re-write each situation in Exercise 1 as a product of two numbers to find the total cost with tax?

   a) $1.05 \times $21.00$
   b) $1.07 \times $45.90$
   c) $1.06 \times $67.50$
   d) $1.08 \times $299.99$
   e) $1.04 \times $39.95$

19. Since the method in #17 works for sales tax (something you are adding on to a price), it should also work for discounts with a slight variation.

   a) What would be different in setting up the problem in part a of #20 if we were taking off a 5% discount of the $21.00 sale price?

   b) Write an expression that is the product of two numbers to solve this problem.

       a) The difference would be we are subtracting 5%, $21.00 – (.05 \times $21.00) = .95 \times $21.00$
       b) $.95 \times $21.00$

Problem 3.1

20. As a promotion, Bill’s bike shop is having a “we’ll pay for the tax” sale. Since it is illegal for Bill not to charge sales tax, his employees come up with a clever way to take the tax off the bill. Bill charges 6% sales tax on each purchase, so Bill decides to give each person a 6% discount. Which of the following is true?

   a) Does this method work? In other words, will the starting sale price end up being the same after the discounts and tax?

   b) Does it matter if the discount is applied first and then the tax, or vice versa? Explain.
a) Students may be surprised that the method in part (a) does not work. For example on a $100 item, the discount would be $6.00, but then the tax would be charged on the $94.00, not on the original $100.00. So the final sale price is $99.64.

b) In fact, it does not matter which method is applied first. This can be noticed if we start with a regular price of P. Doing the discount then the tax we have the expression 1.06 x (.94P) = .9964P. Doing the tax then the discount we have .94 x (1.06P) = .9964P. It may seem clearer once you are able to write the expression as the product of three values and then notice the values commute.

**Problem 3.2**

For 20-25, estimate the solution for each of the following division problems

21. $1 \frac{2}{5} \div \frac{3}{4}$
   a) less than 1  
   b) between 1 and 2  
   c) between 2 and 3  
   d) greater than 3

   b is the correct answer. $1 \frac{2}{5} = \frac{7}{5}$, and $1 \frac{2}{5} < 1 \frac{3}{4}$ so the quotient would be between 1 and 2.

22. $10 \div 1 \frac{7}{8}$
   a) less than 1  
   b) between 1 and 5  
   c) between 5 and 10  
   d) greater than 10

   c is the correct answer. $10 \div 2 = 5$, and $1 \frac{7}{8} < 2$ so the quotient would be slightly greater than 5.

23. $5 \frac{9}{10} \div 1 \frac{1}{2}$
   a) less than 1  
   b) between 1 and 4  
   c) between 4 and 12  
   d) greater than 12

   b is the correct answer. $6 \div 1 \frac{1}{2} = 4$, and $5 \frac{9}{10} < 6$ so the quotient would be between 1 and 4.

24. $14 \frac{2}{7} \div \frac{8}{10}$
   a) less than 1  
   b) between 1 and 7  
   c) between 7 and 14  
   d) greater than 14

   d is the correct answer. $14 \div 1 = 14$, and $14 \frac{2}{7} > 14; \frac{8}{10} < 1$, so the quotient would be greater than 14.

25. $\frac{3}{4} \div \frac{7}{8}$
   a) less than 1  
   b) between 1 and 2  
   c) between 2 and 8  
   d) greater than 8

   a is the correct answer. $\frac{3}{4} \times \frac{7}{8} < \frac{7}{8}$, and $5 \frac{9}{10} < 6$ so the quotient would be between 1 and 4.

26. $\frac{19}{20} \div \frac{6}{10}$
   a) less than 1  
   b) between 1 and 2  
   c) between 2 and 10  
   d) greater than 10

   b is the correct answer. Although common denominators would yield the exact quotient $\frac{19}{12}$, $\frac{6}{10}$ is slightly less than 1, and $\frac{6}{10}$ is slightly greater than one half. Thus, the answer should be greater than 1, but certainly less than 2.

**Investigation 3**
Problem 3.2

27. Use the model below to help answer the questions about how far Danny walked. Suppose Danny walks at a constant rate, and in 45 minutes he was able to walk 2 \( \frac{1}{4} \) miles.

<table>
<thead>
<tr>
<th>Hours</th>
<th>0</th>
<th>( \frac{1}{4} )</th>
<th>1</th>
<th>1 ( \frac{1}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miles</td>
<td>0</td>
<td>2 ( \frac{1}{4} )</td>
<td>3</td>
<td>3 ( \frac{3}{4} )</td>
</tr>
</tbody>
</table>

a) How far did Danny walk in 15 minutes?
   Using the diagram, if you divide \( \frac{1}{4} \) by 3, you would get 15 minutes = \( \frac{1}{4} \) hour. Dividing 2 \( \frac{1}{4} \) = \( \frac{9}{4} \) miles by three results in \( \frac{1}{4} \) mile. Danny walked \( \frac{3}{4} \) mile.

b) How far did Danny walk in 1 hour?
   Because we are assuming Danny walks at a constant rate, he would \( 4 \times \frac{3}{4} = 3 \) miles in 1 hour.

c) How long would it take Danny to walk 4 \( \frac{1}{2} \) miles?
   Since Danny is walking at a rate of 3 miles per hour, it would take him 90 minutes to walk 4 \( \frac{1}{2} \) miles.

d) How long would it take for Danny to walk 3 \( \frac{3}{4} \) miles?
   1 hour and 5 minutes or \( 1 \frac{1}{12} \) hours. By partitioning the segment between 3 and 3 \( \frac{3}{4} \) miles into three pieces, the hours segment between 1 and 1 \( \frac{1}{4} \) = 1 \( \frac{3}{12} \) is also partitioned into three equal pieces.

Problem 3.2

In exercises 27-30, solve each proportion problem.

28.

\[
\frac{4}{5} = \frac{x}{1 \frac{1}{2}} = \frac{\frac{8}{10}}{\frac{3}{2}} = \frac{8}{10} \cdot \frac{2}{3} = \frac{16}{30} = \frac{8}{15}
\]

x = 6, the fraction on the left side is equivalent to \( \frac{4}{1} \), so \( x = 4 \cdot 1 \frac{1}{2} \)

29.

Investigation 3
\[
\frac{5/6}{2/3} = \frac{x}{4/9}
\]

\(x = \frac{5}{9}\), rewriting \(2/3\) as \(4/6\) implies that the left fraction is equivalent to \(5/4\). This means that \(x\) should be \(5/9\).

30.

\[
\frac{6/5}{6/10} = \frac{x}{1^{2/10}}
\]

\(x = 2^{4/10}\), students might notice that the scale factor of the denominators is \(2\) (\(2 \cdot 6/10 = 1^{2/10}\)). This means that \(x\) should be the product of \(2\) and \(6/5\) or \(12/5 = 2^{4/10}\).

31.

\[
\frac{2/3}{5/6} = \frac{x}{5/6}
\]

\(x = 5\), the left fraction is equivalent to \(6\), and \(5 ÷ \frac{5}{6} = 5\). Alternatively, the scale factor of the denominators is \(2^{1/2}\).

**ACE Connection for 3.2**

32. This table shows how to convert liters to quarts.

   a. About how many liters are in 5.5 quarts?

   There are \(1 ÷ 1.06 ≈ 0.94\) L per quart, so 5.5 qt is \(5.5 \times 0.94 ≈ 5.17\) L.

   b. About how many quarts are in 5.5 liters?

   In 5.5 L there are \(5.5 \times 1.06 = 5.83\) quarts.

   c. Write an equation for a rule that relates liters \(L\) to quarts \(Q\).

   From the unit rates, \(Q = 1.06L\) and \(L = 0.94Q\).

**Problem 3.3**

Ming has many other primates that she works with at the zoo, and each one has slightly different dietary needs for their mixes of high fiber and high protein mix. For exercises 32-34, determine the ratio of high protein to high fiber mix, then answer the questions.

**Investigation 3**
33. Orangutan mix

<table>
<thead>
<tr>
<th>High Protein Mix</th>
<th>21</th>
<th>24</th>
<th>27</th>
<th>18</th>
<th>33</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Fiber Mix</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>6</td>
<td>11</td>
</tr>
</tbody>
</table>

a) Write an equation that relates the number of scoops of high protein mix to high fiber mix.
   \[ P = 3F \text{ or } P \div 3 = F \]

b) If Ming mixes 12 scoops of high protein mix, how many scoops of high fiber mix would she mix?
   \[ 4, \text{ substitute 12 for } P, \text{ and solve for } F. \]

c) For every 1 scoop of high protein mix, how many scoops of high fiber mix would Ming need?
   \[ \frac{1}{3} \text{ scoop.} \]

d) Ming would like to make a graph so that she can more quickly determine the mix ratios. Make a graph for Ming with the high protein mix on the y-axis, and high fiber mix on the x-axis.

Problem 3.3

34. The ratio of high fiber mix to high protein mix for baby gorillas is 30% to 70%.

a) What is the unit rate for this mixture?
   \[ \frac{3}{7} \approx 0.43 \text{ or } \frac{7}{3} = 2.333\ldots \]

b) Fill in the rate table below

<table>
<thead>
<tr>
<th>High Protein Mix</th>
<th>14</th>
<th>1</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Scoops)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>High Protein Mix</th>
<th>7</th>
<th>14</th>
<th>1</th>
<th>( \frac{7}{3} )</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Scoops)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>High Protein Mix</th>
<th>3</th>
<th>2</th>
<th>( \frac{3}{7} )</th>
<th>1</th>
<th>( \frac{7}{7}x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Scoops)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c) Graph the relationship between high protein mix and high fiber mix.
d) Use the table and graph to write an equation relating the two variables.

\[ P = \frac{7}{3} F \text{ or } F = \frac{3}{7} P \]

35. Ming was given the following graph to show the mix ratio for the adult baboons at the zoo.

a) What is a good estimate for the number of scoops of high protein mix Ming should use with 5 scoops of high fiber mix?

About 4 scoops

b) To help Ming remember the ratio easier she would like to have a ratio that uses small whole numbers. What ratio would be good for Ming to remember?

5 scoops of high protein mix: 6 scoops of high fiber mix.

c) Ming wants to be more precise with her mixture so she decides to write an equation based on the graph. Write the equation that you think Ming should use.
Investigation 3

\[ P = \frac{5}{6} \]

\[ P_{\text{blue}} = \frac{7}{10} \]  

If Ming mixed 45 scoops of high protein mix, how many scoops of high fiber mix should she use?

54 scoops.

Extensions

Problem 3.1

36. The city of Spartanville runs two summer camps—the Green Center and the Blue Center. The table below shows recent attendance at the two camps.

<table>
<thead>
<tr>
<th></th>
<th>Green</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>125</td>
<td>70</td>
</tr>
<tr>
<td>Girls</td>
<td>75</td>
<td>30</td>
</tr>
</tbody>
</table>

In this exercise, you will show how several approaches can be used to answer the following question. Which center seems to offer a camping program that appeals best to girls?

a. What conclusion would you draw if you focused on the differences between the numbers of boy and girl campers from each center?

The camps were relatively close in terms of the difference between boys and girls. The difference between the two camps was that 45 more girls attended Camp Green than Camp Blue and 55 more boys attended Camp Green than Camp Blue. Therefore, one could conclude that Camp Green appeals best to girls. There were 50 more boys than girls at Camp Green and 40 more boys than girls at Camp Blue.

b. How could you use fractions to compare the appeal of the two centers’ camping programs for boys and girls? What conclusion would you draw?

You could compute the fraction of boys to total numbers of campers at each camp and the fraction of girls to total numbers of campers at each camp. The total for Camp Green is 125 + 75 = 200. The fraction of boys at Camp Green is then \( \frac{125}{200} = \frac{5}{8} \) and the fraction for girls at Camp Green is \( \frac{75}{200} = \frac{3}{8} \). The total for Camp Blue is 70 + 30 = 100. The fraction of boys at Camp Blue is \( \frac{70}{100} = \frac{7}{10} \), and the fraction for girls at Camp Blue is \( \frac{30}{100} = \frac{3}{10} \). One can then compare fractions with like denominators, comparing \( \frac{5}{8} = \frac{25}{40} \) for boys at Camp Green to \( \frac{3}{8} = \frac{15}{40} \) for girls at Camp Green. \( \frac{7}{10} \) and \( \frac{3}{10} \) for Camp Blue become \( \frac{28}{40} \) boys and \( \frac{12}{40} \) girls for Camp Blue. One could then conclude that more girls prefer Camp Green and more boys prefer Camp Blue based on fractions of the total campers.

c. How could you use percents to compare the appeal of the two centers’ camping programs for boys and girls? What conclusion would you draw?

62.5% of campers at Camp Green were boys and 70% of campers at Camp Blue were boys. A conclusion could be that boys preferred Camp Blue to Camp Green. 37.5% of campers at Camp Green were female and 30% at Camp Blue were female. Girls preferred to attend Camp Green over Camp Blue.

d. How could you use ratios to compare the appeal of the two centers’ camping programs for boys and girls? What conclusion would you draw?

The ratio of 5 to 3 describes boys to girls at Camp Green and a ratio of 7 to 3 describes boys to girls at Camp Blue. The ratio of boys to girls is greater at Camp Blue than Camp Green.

Problem 3.2 (Inv. 2)

37. Use the table below.
Investigation 3

<table>
<thead>
<tr>
<th>Sport</th>
<th>Girls</th>
<th>Boys</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basketball</td>
<td>30</td>
<td>80</td>
</tr>
<tr>
<td>Football</td>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>Soccer</td>
<td>120</td>
<td>85</td>
</tr>
<tr>
<td>Total Surveyed</td>
<td>160</td>
<td>225</td>
</tr>
</tbody>
</table>

a. In which sport do boys most outnumber girls?
   Football (The ratio of boys to girls is 6:1, the greatest ratio of all the sports.)

b. In which sport do girls most outnumber boys?
   Soccer

c. The participation in these team sports is about the same for students at Key Middle School.
   
   i. Suppose 250 boys at Key play sports. How many would you expect to play each of the three sports?
      Rounded to the nearest whole number: Basketball = 89, football = 67, soccer = 94.
   
   ii. Suppose 240 girls at Key play sports. How many would you expect to play each of the three sports?
      Rounded to the nearest whole number: Basketball = 45, football = 15, soccer = 180.

Mathematical Reflections

1. What are some strategies you have developed for solving proportions?

   Students may mention scaling up. They should have some efficient way of deciding on the scale factor to use. They may use a common denominator variation, where both ratios in the proportion are scaled up to have a common denominator. Some students may like to reduce one of the ratios to a single number, by dividing (making a unit rate).

2. Describe a strategy for converting a rate measured in one pair of units to a rate measured in another pair of units, for example, ounces per cup to pounds per gallon, or calories per pound to calories per kilogram.

   Suppose we want to rewrite a rate of 200 calories per pound as C calories per kilogram. We need to know that 1 kilogram is 2.2 pounds; that is a unit rate. Then we can set up a proportion:

   200 calories: 1 pound = C calories: 2.2 pounds.

3. How are the ideas about scaling that you used in Stretching and Shrinking the same as or different from the ideas about proportions and rates you used in Comparing and Scaling?

   In Stretching and Shrinking we found scale factors by comparing a length on the original figure to the corresponding length on a scaled up (or down) figure. This involves dividing the two lengths to get the factor. Then we used that scale factor to find the unknown length.
This is just the same as finding the scale factor to write an equivalent ratio and solve a proportion in *Comparing and Scaling*. The scale factor we found in *Stretching and Shrinking* is also just like a unit rate: for example, say the scale factor is 1.8, then for every 1 unit length on the original figure the length on the scaled up figure is 1.8 units.

4. What are some connections you have found among unit rates, proportions and rate tables?

A unit rate is found in the rate table by scaling one quantity to 1. You can achieve this unit rate by dividing a related pair of quantities in a rate table.

A unit rate also appears on one side of a proportion if you divide the two parts of the ratio that is known.

Proportions are made of equal ratios. Rate tables show pairs of equal ratios or rates.
Comparing and Scaling, Paper Pool Project
Teacher's Guide

TASK:
For this project, students are asked to play a game called “paper pool”. The game is played on rectangular square grid tables. An imaginary ball is hit from the lower left-hand corner marked A, at a 45° angle. A ball hit in this way will bounce off the sides at a 45° angle. The ball continues to roll until it hits a pocket. Pockets are located at each of the corners of the table. Students play paper pool on different sized rectangular tables. The task asks students to predict in which pocket the ball will stop and how many hits (anything making contact with the ball – the sides of the table, the imaginary cue, the pocket) will occur by the time the ball comes to a stop (the ball reaches a pocket).

To do the task, students will need to investigate several different sized paper pool tables. They will need to gather and organize data and search for patterns. Finding a solution to the task will require students to recognize relationships between rectangles whose sides have the same ratio. We recommend that the project be started near the end of the unit.

MATERIALS:
grid paper
paper pool labsheets
colored pencils or markers

GOALS FOR STUDENTS:
• organize data and look for patterns
• recognize rectangles whose sides have the same ratio (similar rectangles)
• use simplest ratio to predict stopping pocket and number of hits.

Launch
Read through the task introduction with your students. Make sure they understand how the ball travels on the paper pool tables and how you count the number of hits that occur on any table. Check to see that students have drawn the paths correctly for the two sample tables.
An extension question is offered with this task. You could assign this to all or use it as an extra challenge for those groups that want to further investigate patterns.

**Explore**

We recommend that students work on this project with a partner. One class period will be needed for pairs to work together and collect their data. Students can continue to investigate the task and draft their reports outside of class. You may want to use part of a second class period for groups to compare results and finalize reports.

**Summarize**

You may want to have groups share their results. If the extension was given as an extra challenge, be sure to have any group who attempted it share their results.

Comparing and Scaling, Appendix B

Scored Paper Pool Project with Teacher Comments

For this project I had my students work in pairs but each of them had to write their own report. I used the suggested time schedule; one class period and then half of a class period three days later. This allowed students who needed more time to investigate the situation and look for patterns to do so at home. The half period of class was used to discuss and revise reports as partners shared what each had written. Most of my students found this project interesting and were very engaged in the mathematical investigation involved in this project.

The students’ reports came in several forms and levels of quality. The two projects below are examples of student work from the class. Because of the amount of space, I have not included the students’ labsheets or any additional drawings they did. The work was scored using the CMP Holistic by Category rubric provided in the assessment package. An explanation of the scores for each of the two projects is written after each example.

**ADD MARY BETH’S WORK**

Mary Beth’s project received 8 of the 8 points for the mathematics that she presented in her report. A four was given for her sophisticated rule on which corner the ball would stop. Her rule for the stopping corner covers all possible cases. She notes it makes a difference if the table is odd by even or even by odd but does not tell in her description if she is giving the bottom
dimensions first or the side length. Her drawings and organizational tools (she made tables to organize her information) made it possible to determine what she meant, so full credit for her rules was given. A four was given for her sophisticated rule for the number of hits that would occur. Her rule identifies the sum of the dimensions of the table as the needed relationship. This is the correct relationship that allows you to find out the total number of hits that will occur, but her written report does not state that it is the sum of the dimensions when expressed as a ratio in simplest form (or what she calls earlier “basic” form). I might have counted down for this if it were not for the fact that her labsheets, new drawings, and organizational tools showed that she understood that it was the sum of the simplified ratio.

A score of four was given to Mary Beth’s work for problem solving and reasoning and a 3 was given for communication. Her reasoning for her rules was complete when you took into account her written summary, new drawings, and organizational tools. The 3 was given for communication because of the effort that the reader needed to make to sort out which side she was referring to for her odd by even and even by odd rules. Also decision was make because of the lack of clarity and completeness in her written description of her rule for stopping corner.

Mary Beth’s new table, labsheet, and organizational tool were included in her report and she was given all the points for this section due to their quality and completeness. Mary Beth has 20 of the 21 points for this project and was given an A for the project.

**MATHEMATICS -- ending corner rule**

<table>
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<tr>
<th></th>
<th>4 out of 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>ending corner rule</td>
<td></td>
</tr>
<tr>
<td>total hits rule</td>
<td></td>
</tr>
</tbody>
</table>

**PROBLEM SOLVING & REASONING**

<table>
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<tr>
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**COMMUNICATION**

<table>
<thead>
<tr>
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**CHECKLIST**

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<tr>
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</tr>
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<tr>
<td>new tables</td>
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<tr>
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<tr>
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**TOTAL POINTS**

<table>
<thead>
<tr>
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**ADD HEATHER’S WORK**

Heather’s project received 5 of the 8 points for the mathematics that she presented in her report. A 3 was given for six basic rules and one sophisticated rule that identified at which corner the ball would stop. A two was given for her three basic rule for the number of hits that would occur (a fourth rule “two hits” is started but not finished). Her rules for the stopping corner cover
several possibilities and her rule of, “On an odd by odd it will always end up at corner C”, is considered a sophisticated rule. She does not address the orientation of the rectangular and the reader can only make sense of her rules by examining her drawings and organizational tools. Heather’s rules for stopping corner suggest that she looked for patterns. Her rules for the number of hits are part of some of her rules for ending corners. The count she gives for the number of hits is incorrect and suggests that she does not understand what counts as a hit. It seems that she has not counted the hit from the imaginary cue nor the hit at the last pocket in her count.

A score of 2 was given to Heather’s work for both communication and problem solving and reasoning. Because she shows no evidence of being able to reason about how many hits will occur and because the reader’s only evidence of Heather's reasoning is through her labsheet and single organization chart, she was given a 2 for problem solving and reasoning. A 2 was given for communication because the reader needs to make a significant effort to follow the student’s report. Because Heather does not deal with orientation of the rectangles, one must make an effort to sort through her work and make sense of the rules she has given.

Heather’s new tables to demonstrate her rules were complete and she was given 2 point for her work on this. Her labsheet was also complete, thus she received the one point for including these papers in her report. She received a 1 for her organizational tools. Her tools only included only a table/chart that organized the information as to which pocket the ball would come to a stop. She did not have include any organizational tool that addressed the number of hits that occurred.

Heather’s work was given 13 of 21 points and a grade of a C in my class. Heather's labsheet shows that she is not counting the number of hits correctly. I'm not sure why this is so because when we launched the project in class both she and her partner were able to get the correct number of hits for the paper pool tables in the launch. Further instruction will probably be needed to help the two of them address the issue of hits. I will also want to talk to Heather and all other students about how to look for patterns and how to organize information to help in looking for patterns.

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Holistic by Category Scoring Rubric for Comparing and Scaling Paper Pool Project

NOTE: The scoring rubric is for the project without the extension question assigned.

Mathematics (0-8 points total)

Rules/Patterns for predicting the ending corner (0 -4 points)

0  Student did not engage, no patterns or rules are given

1  Student shows evidence of searching for a pattern but states no original pattern or rule OR students states one specific rule.

2  Student states at least two correct specific rules.

3  Student states a correct sophisticated rule and/or several specific rules which address several possible situations for where the ball will stop.

4  Student states at least one correct sophisticated rule and addresses all possible situations for which corner the ball will stop.

Rules/Patterns for predicting the total number of hits (0 -4 points)

0  Student did not engage, no patterns or rules are given

1  Student shows evidence of searching for a pattern but states no original pattern or rule OR students states one correct specific rule.

2  Student states at least two correct specific rules.

3  Student states a correct sophisticated rule and/or several specific rules which address several possible situations for the number of hits that will occur.

4  Student states a least correct one sophisticated rule and addresses all possible situations for the number of hits that will occur.
**Problem Solving and Reasoning (0 to 4 points)**

0  Student does not engage in the given task

1  Student shows reasoning about rules through words or organizational instruments used but it may be faulty -- incorrect logic or non-sensible statements in the context of the problem OR only reasons through one specific rule.

2  Student shows reasoning about rules through words or organizational instruments but reasoning is weak -- tests an inadequate number or variety of situations drawing conclusions that would require testing more cases or examining more varied arrangements OR has only one or two specific rules and does not address both situations.

3  Student shows adequate reasoning to support given sophisticated rule(s) or gives complete reasoning to support specific rules for both situations.

4  Student shows complete reasoning to support sophisticated rules for both situations.

**Communication (0 to 4 points)**

0  Student does not communicate in any form.

1  Student does not address the task presented.

2  Significant effort is needed to follow the student’s report.

3  With some extra effort, the reader can follow the student’s report.

4  Report is clearly stated and easy to follow.
Checklist (5 points possible)

2 If student gives a correct new table for each rule given. Must have at least two rules.
   (One rule and one correct new table that fits the rule is worth 1 point.)

1 Student does labsheets for paper pool.

2 Students uses organizational tool(s) to search for patterns and rules for the task.
   Quality is the determining factor for giving a paper 0, 1, or 2 points.
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Looking Back and Looking Ahead

Unit Review

1. There are 300 students in East Middle School. To plan transportation services for the new West Middle School, the school system surveyed East students. The survey asked whether students ride a bus to school or walk.
   - In Mr. Archer’s homeroom, 20 students ride the bus and 15 students walk.
   - In Ms. Brown’s homeroom, 14 students ride the bus and 9 students walk.
   - In Mr. Chavez’s homeroom, 20 students ride the bus and the ratio of bus riders to walkers is 5 to 3.

   a. In what ways can you compare the number of students in Mr. Archer’s homeroom who are bus riders to the number who are walkers? Which seems to be the best comparison statement?

   **Difference:** this is done by subtracting the number of walkers from the number of bus riders, which answers the question of “How many more ride the bus than walk?” For Mr. Archer’s room the difference is 5.

   **Ratio:** can also be used to compare the number of bus riders to the number of walkers. For Mr. Archer’s room, the ratio is 20:15 or 4:3.

   **Fractions:** done by writing the number of walkers out of the total students from the room and then the number of bus riders out of the total students from the room and comparing. For Mr. Archer’s room, the fractions are 4/7 bus riders and 5/7 walkers.

   **Percent:** can also be used to determine the number of bus riders and walkers out of 100. For Mr. Archer’s room, the percent of bus riders is 57% and the percent of walkers is 43%.

   **Unit rate or scaling:** can be used by dividing the number of walkers into the number of bus riders and determining that there are 1.3 bus riders for every walker, or the number of bus riders is 1.3 times the number of walkers.

   The best statement would probably be the ratio, as you are trying to compare two parts, bus riders to walkers.

   b. In what ways can you compare the numbers of bus riders and walkers in Ms. Brown’s homeroom to those in Mr. Archer’s homeroom? Again, which seems the best way to make the comparison?

   One could compare the number of bus riders and walkers between homerooms through use of percents. Percents allow comparison of both bus riders and walkers to be out of 100, so that the difference...
in total students between the homerooms would not matter. Mr. Archer’s homeroom had 57% bus riders and 43% walkers as compared to 61% bus riders and 39% walkers from Ms. Brown’s. The ratio of bus riders to walkers can be compared between rooms, such as 4 to 3 for Mr. Archer’s and 14 to 9 for Ms. Brown’s (or 12 to 9 for Mr. Archer’s and 14 to 9 for Ms. Brown’s). A comparison can also be made by the unit rate or scale factor, so determine the number of walkers per bus ride or bus riders per walkers for each homeroom; 1.3 bus riders per walker for Mr. Archer and 1.6 bus riders per walker for Mr. Brown. The difference between bus riders and walkers could also be compared between homerooms. The difference is 5 for Mr. Archer’s and 5 for Ms. Brown’s. The best method seems to be percent, as the number of students in each homeroom is not the same and therefore to be able to make a comparison on unlike quantities, percents work the best because it places the numbers in amounts “out of 100”.

c. How many students in Mr. Chavez’s homeroom walk to school?
   12. using equivalent ratios, $\frac{5}{3} = \frac{20}{?}$. The scale factor is 4.

d. Use the information from these three homerooms. About how many East Middle School students would you expect to walk to school? How many would you expect to ride a bus?
   180 bus riders and 120 walkers. The total bus riders from the 3 homerooms is 54 and the total walkers from the 3 homerooms 36. The total number of students in the 3 homerooms is 90. Using equivalent fractions for bus riders, $\frac{54}{90} = \frac{?}{300}$. The scale factor is $3 \frac{1}{3}$.

e. Suppose the new West Middle School will have 450 students and a ratio of bus riders to walkers that is about the same as that in East Middle School. About how many West students can be expected in each category?
   270 bus riders and 180 walkers. The ratio of bus riders to walkers at East is 3 to 2.

2. The Purr & Woof Kennel buys food for animals that are boarded. The amounts of food eaten and the cost for food are shown below.

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CREATIVE ART
Bag of cat food and dog food
5143
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a. Is cat food or dog food cheaper per pound?
Dog food. Dog food costs 49 cents per pound and cat food costs 60 cents per pound.

b. Is it cheaper to feed a cat, a small dog, or a large dog?

A cat is cheapest to feed. The cost per pound of cat food is 60 cents and a cat eats 1/3 pound a day; therefore, it costs 20 cents a day to feed a cat. Dog food costs 49 cents per pound. A small dog eats ½ pound a day and therefore costs about 245 cents a day, while a large dog eats 1 ¼ pounds a day and therefore costs about 61 cents a day to feed.

c. On an average day, the kennel has 20 cats, 30 small dogs, and 20 large dogs. Which will last longer: a bag of cat food or a bag of dog food?

Cat food. For cat food, 20 x (1/3) = 6 2/3 lb per day. Since the bag holds 10 lb, it will last 10 ÷ 6 2/3 = 1.5 days. For dog food, 30 x (1/2) + 20 x (1 ¼) = 40 lb/day used. Since the bag holds 50 lb, it will last 50 ÷ 40 = 1.25 days.

d. How many bags of dog food will be used in the month of January? How many bags of cat food will be used?

About 20.67 bags of cat food (21 opened) and 24.8 bags of dog food (25 opened). There are 31 days in January. Since 1 bag of dog food lasts 1.25 days, 24.8 bags will be needed. Since 1 bag of cat food lasts 1.5 days, 20.67 bags will be needed.

e. The owner finds a new store that sells Bow Chow in 15 pound bags for $6.75 per bag. How much does that store charge for 50 pounds of Bow Chow?

The 5—lb bag will cost $22.50. At the new store, the cost of Bow Chow per pound is 45 cents ($0.45). Therefore, for 50 lb, it will cost $0.45 x 50 = $ 22.50.

f. Which is a better buy on Bow Chow: the original source or the new store?

The new store is a better deal. It costs $22.50 for a 50-lb bag, but at the old store it costs $24.50. Therefore, the owner will save $2 on a 50-lb bag if he/she shops at the new store.
Explain Your Reasoning

Answering comparison questions often requires knowledge of rates, ratios, percents, and proportional reasoning. Answer the following questions about your reasoning strategies. Use the preceding problems and other examples from this unit to illustrate your ideas.

3. How do you decide when it makes sense to compare numbers using ratios, rates, or percents rather than by finding the difference of the two numbers?

You need to look at what the question is asking. Ratios are used when you are comparing two quantities, such as two parts of a mix and want to know the scale between them. Percentages are used when you compare two things in different amounts. Differences are used when you talk about a discrepancy between two amounts. Rates are used when you want to talk about a direct comparison between two sets. When you want to compare number of girls to number of boys, a ratio would be appropriate. When you want to know how much bigger or farther one thing from another, using differences is appropriate. When you want to know the number of something per some other unit, such as the number of calories per cookie, then a rate would be appropriate. When you want to know how many people out of 100, then a percentage would be appropriate.

4. Suppose you are given information that the ratio of two quantities is 3 to 5. How can you express that relationship in other written forms?

3 to 5 can also be expressed as 3:5. One can express it using the fraction 3/5, or the percent 60%. 3 to 5 can also be expressed as 0.6 per unit.

5. Suppose that the ratio of two quantities is 24 to 18.
   a. State five other equivalent ratios in the form “p to q.”
      12 to 9, 4 to 3, 240 to 180, 8 to 6, 48 to 36.
   b. Use whole numbers to write an equivalent ratio that cannot be scaled down without using fractions or decimals.
      4 to 3.

6. What strategies can you use to solve proportions such as \( \frac{5}{8} = \frac{?}{24}\) and \( \frac{5}{?} = \frac{24}{8}\)?

Find the scale factor between the two given numbers in the same numerator or denominator location and then multiply by the scale factor. For example, to solve \( 5.8 = 12/\?\), find the scale factor between 12 and 5 \((12 \div 5 = 2.4)\), then multiply 8 by 2.4, which is 19.2. to solve, \( 5/8 = \?/24\), find the scale factor between 24 and 8 \((24\div 8 = 3)\) and multiply 5 by the scale factor \((15)\). Strategies include finding the scale factor or equivalent fractions.

7. How does proportional reasoning enter into the solution of each problem?
a. You want to prepare enough of your favorite recipe to serve a large crowd.

With the recipe, you scale up or down the quantity of ingredients needed based on the numbers of people you need to serve and the number of people the recipe can feed in original form. To change recipes to meet the number of people, you would need to multiply the amount of each ingredient by the ratio of the number of people you want to feed the number of people you can feed per the original recipe. For example, to scale up a recipe that serves 4 to serve 12, multiply 3 by each quantity of ingredient listed. Similarly, to scale down a recipe that serves 20 to serve 4, multiply each quantity of ingredient by 4/20 (or 1/5).

b. You want to use the scale of a map to find the actual distance between two points in a park from their locations on the map.

Compare the difference on the map and use the scale factor given in the legend to determine the actual distance (multiply the difference you find on a map using a ruler by the ratio/scale factor given on the map). For example, the distance between two cities on a map is 0.25 inches. If the scale factor is 1 inch = 100 miles, then the actual distance is 25 miles.

c. You want to find which package of raisins in a store is the best value.

Use proportion to determine the cost per unit of weight (ounces or pounds). Then the unit rates of cost per weight can be compared to determine which is better deal or more economical.

d. You want to use a design drawn on a coordinate grid to make several larger copies and several smaller copies of that design.

You can multiply the coordinates by scale factors to make larger or smaller copies of the design. The copies would then be similar to the original. The scale factor would be the ratio of one side of the design to the corresponding side of a similar figure of the design.