The Role of Problem Solving in High School Mathematics

Reaching All Students

Problem solving has been the focus of a substantial number of research studies over the past thirty years. It is well beyond the scope of this paper to even attempt to summarize this body of research. Those interested in significantly broader reviews of research related to problem solving should see Schoenfeld (1985), Charles & Silver (1988), and Lesh & Zawojewski (2007). This paper focuses on the most recent research related to problem solving that has a direct impact on the way mathematics is taught every day in secondary mathematics classrooms.

Learning Mathematics: The Traditional Role for Problem Solving

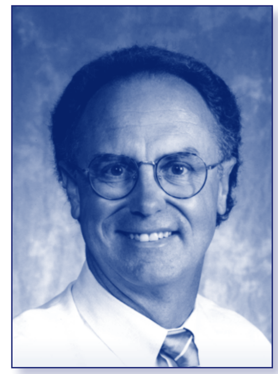
Problem solving has always played an important role in learning mathematics. *An Agenda for Action* (NCTM, 1980) said, "Problem solving [should] be the focus of school mathematics. . . " In 2001 the National Research Council (Kilpatrick, J., Swafford, J. & Findell, B. 2001) reaffirmed the importance of problem solving by identifying it as one of five strands of mathematical proficiency (see Figure 1).

Figure 1: Five strands of mathematical proficiency (NRC, 2001)

- Conceptual understanding
- Procedural fluency
- Problem-solving competence
- Reasoning
- Helpful attitudes and beliefs about mathematics

While problem solving has always had a role in learning mathematics, its role has evolved over the years. The oldest role that prblem solving has and continues to have in learning mathematics is that of a context for practicing and applying concepts and skills. This role has been referred to as "teaching FOR problem solving." In this role, concepts and skills are developed and then real-world problems, usually called "applications," are presented where students must choose and apply appropriate concepts and skills to find solutions.

A clear finding from research related to teaching FOR problem solving is that practice solving real applications improves students' problem-solving abilities



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IF these applications are of sufficient variety and complexity that thinking is required to understand them and to identify the relevant concepts and skills needed to solve them. In other words, the applications must be real *problems* for students. Applications that require little or no thinking about the concepts or skills needed to solve them are called *exercises*, not problems. Applications presented as exercises do little to improve students' problem–solving abilities.

Learning Mathematics: A New Role for Problem Solving

Mathematics makes sense to students and is easier to remember and apply when students understand the mathematics they are learning. Also, students who understand mathematical concepts and skills more readily learn new mathematical concepts and skills. Students who learn mathematics with understanding feel a real sense of accomplishment and thus are motivated to learn more mathematics and to succeed in mathematics. Students who understand mathematics become autonomous learners of mathematics.

"Introducing concepts and skills in problem-solving contexts evokes thinking and reasoning about mathematical ideas."

Research has shown that understanding is best developed through a balance of (a) *introducing* concepts and skills in the context of solving problems and (b) presenting examples to students in the context of a problem-focused and question-driven classroom conversation. Developing mathematical understanding in these ways is called "teaching THROUGH problem solving."

Introducing Concepts and Skills Through Problem Solving

One of the strongest research findings in the past ten years is that problem solving plays a critical role in the initial *learning* of mathematical concepts and skills, not just as a context for *practicing* concepts and skills as discussed above. Research shows that understanding develops during the process of solving problems in which important math concepts and skills are embedded (Schoen & Charles, 2003). Introducing concepts and skills in problem-solving contexts evokes thinking and reasoning about mathematical ideas. Students who think and reason about mathematical ideas learn to connect these new ideas to ideas previously learned, that is, they develop understanding.

The task shown in Figure 2 is a problem that can be used to introduce *point-slope form* for linear equations. Prior to this task, students had learned to write and graph equations using the *slope-intercept form* of a linear equation. In this task, they are not given the slope or the y-intercept. From the graph shown, students can solve this problem in different ways. One way is to estimate the y-intercept and use any two points to find the slope. Another way is to find the slope using any two points and then substitute the slope and the coordinates of any point given into the slope-intercept form of a linear equation to calculate the y-intercept.

This task and both ways of solving it described above can be used to connect the students' prior learning, slope-intercept form, to the new idea in the lesson, point-slope form. Solving this task illustrates the important idea that the form most easily used to represent a linear equation depends on the information one has about the line (e.g., the slope).

Figure 2. An example of a problem-based task used to introduce point-slope form. (Editor – see Algebra 1 Lesson 5-4)

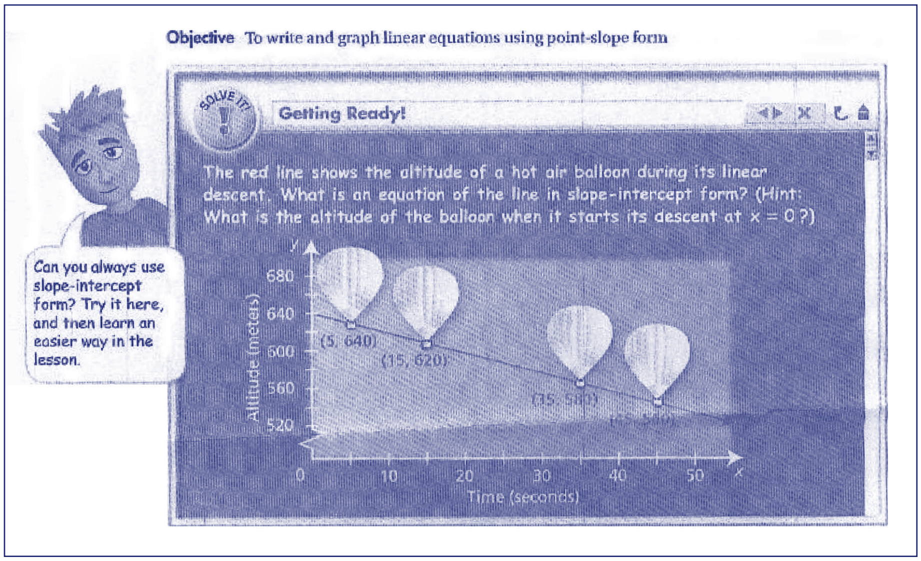
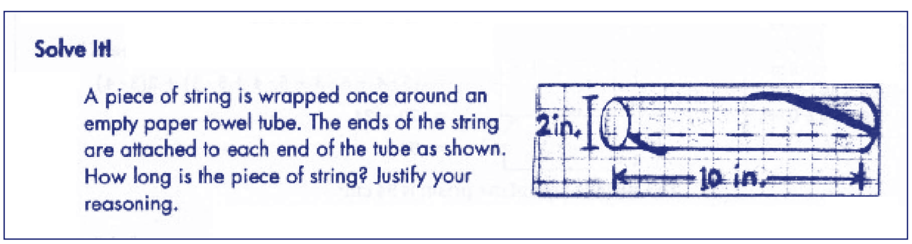


Figure 3 is another example of a problem that can be used to introduce a geometry lesson on *surface areas of prisms and cylinders*. Notice that this problem focuses only a cylinder, not a prism. Problem-based learning tasks used to start lessons need not address all concepts and skills in a lesson. Many high school mathematics lessons contain too many ideas for students to explore all of them through problem-based learning tasks at the start of lessons. The important point is that a problem-based learning task develops an initial understanding of one or more concepts or skills that will be formally taught through the lesson.

Figure 3. An example of a problem-based task used to introduce surface areas of prisms and cylinders. (Editor – see Geometry Lesson 11-2)



Research related to introducing concepts and skills through problem solving has shown that solving problems like the ones in Figures 2 and 3 and discussing alternative solutions promotes understanding IF the important mathematics students were supposed to learn through solving the problem is made explicit. This should happen in two ways. First, after students share and discuss alternative solutions to a problem, the teacher must connect the students' work on that problem to the new concept or skill of focus for the remainder of the lesson by sharing comments that make the connection explicit. For the example in Figure 2, the teacher should comment on the fact that the information given about a line determines how one can find the equation of the line. And, in this case, that although one can use the information given to find the equation of the line in slope-intercept form, there are other forms of linear equations that can also be used depending on the information given. In the remainder of the lesson, the students will learn another way to write an equivalent equation for a line, point-slope form.

The second way to make the important mathematics explicit related to a problem-based task is the next ingredient in teaching mathematics through problem solving – presenting examples through problem-focused classroom conversations.

Presenting Examples Through Problem-Focused Classroom Conversations

Introducing new concepts and skills through problem solving initiates understanding. Following up with the artful presentation of examples to students further develops understanding. Presenting and discussing examples has always been an important part of teaching and learning mathematics. However, we have learned that there are effective and ineffective ways to do this.

"Show and tell" is not an effective instructional approach to present examples where understanding is a goal. That is, the teacher showing students an example and walking them through a sequence of steps with a verbal explanation of what to do does not help most students understand mathematics. Research shows that an effective alternative is for the teacher to introduce examples as though they are problems to be solved, and then have a classroom conversation driven by rich questions focusing on why the various parts or steps in the example make sense. Presenting examples in this way promotes understanding because rich questions focus attention on important elements of the concept or skill and they make explicit the rationale for why these elements make sense. As noted earlier, when concepts and skills make sense to students, they learn faster, they remember better, and they are better able to use concepts and skills in subsequent problem-solving situations.

"Introducing new concepts and skills through problem solving initiates understanding."

Another significant benefit of presenting examples as problems and having question-driven classroom conversations about those problems is that the teachers' questions and statements can model mental habits of thinking and reasoning that promote learning and positively impact performance. Figures 4 and 5 illustrate how examples can be presented as problems and how questions can be asked and comments made that model "thinking" and "planning." Modeling effective mental habits of thinking and reasoning is an efficient and effective way for students to acquire these mental habits. Modeling can happen visually and orally as students watch and listen to the teacher and other students solve problems and it can also happen by students reading illustrations of effective thinking and reasoning as shown in Figures 4 and 5.

Figure 4: An example from Algebra 1 presented as a problem that models planning and thinking.

Plan

How can you visualize the equation?
You can *draw a diagram*. Use a model like the bar diagram below to help you visualize any equation. A model for the equation $x + 13 = 27$ is

27	
x	13

Problem 1 Solving an Equation Using Subtraction

What is the solution of $x + 13 = 27$?

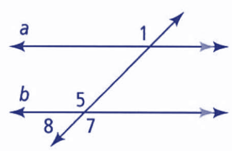
Think	Write
You need to isolate x. Start by writing the equation.	$x + 13 = 27$
Undo addition by subtracting the same number from each side.	$x + 13 - 13 = 27 - 13$
Simplify each side of the equation.	$x = 14$
Substitute your answer into the original equation to check it.	$x + 13 = 27$ $14 + 13 \stackrel{?}{=} 27$ $27 = 27$ ✔

Figure 5: An example from Geometry presented as a problem that models the Know-Need-Plan phases of problem solving.

Proof **Problem 2 Proving an Angle Relationship**

Given: $a \parallel b$

Prove: $\angle 1$ and $\angle 8$ are supplementary.



Know
- $a \parallel b$

From the diagram you know
- $\angle 1$ and $\angle 5$ are corresponding
- $\angle 5$ and $\angle 8$ form a linear pair.

Need
$\angle 1$ and $\angle 8$ are supplementary, or $m\angle 1 + m\angle 8 = 180$.

Plan
Show that $\angle 1 \cong \angle 5$ and that $m\angle 5 + m\angle 8 = 180$. Then substitute $m\angle 1$ for $m\angle 5$ to prove that $\angle 1$ and $\angle 8$ are supplementary.

Statements	Reasons
1) $a \parallel b$	1) Given
2) $\angle 1 \cong \angle 5$	2) If lines are $\parallel$, then corresp. ∡ are ≅.
3) $m\angle 1 \cong m\angle 5$	3) Congruent ∡ have equal measure.
4) $\angle 5$ and $\angle 8$ are supplementary.	4) ∡ that form a linear pair are suppl.
5) $m\angle 5 + m\angle 8 = 180$	5) Def. of suppl. ∡.
6) $m\angle 1 + m\angle 8 = 180$	6) Substitution Property
7) $\angle 1$ and $\angle 8$ are supplementary.	7) Def. of suppl. ∡.

Summary

Developing students' abilities to solve problems will remain a critical goal for secondary school mathematics. Practice solving rich applications of mathematics, application problems not exercises, is a necessary component of a curriculum that improves problem-solving performance. However, problem solving in the secondary mathematics curriculum should not be limited to practice solving application problems. A mathematics curriculum that develops a deep understanding of concepts and skills must be driven by *teaching THROUGH problem solving*. New concepts and skills should be introduced in the context of solving problems that have important mathematical ideas embedded. Then examples should be used that extend understanding and promote thinking and reasoning. Presenting examples as problems and modeling effective thinking and reasoning habits promotes understanding and mastery.

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