1 Sequences and Series

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CHAPTER OVERVIEW

Background

In grade 9, students graphed linear relations and solved linear equations, which is necessary background knowledge for understanding arithmetic sequences and series. In grade 10, these concepts are extended to include linear functions. By the time students enter grade 11, they should be able to determine the slope of the graph of a linear function. This is related to the constant difference in an arithmetic sequence.

In grade 9, students were introduced to powers and roots (limited to whole number exponents) and square roots of perfect squares. In grade 10, these concepts were extended to include integral and rational exponents and to determining $n$th roots. This knowledge is a prerequisite for solving problems involving geometric sequences and series.

Rationale

Sequences and series are introduced in grade 11. Students use their understanding of linear functions to develop the properties of arithmetic sequences and series, then solve related problems. They derive rules for determining the $n$th term of an arithmetic sequence and the sum of the first $n$ terms of an arithmetic series.

Students are introduced to geometric sequences and series, and distinguish them from arithmetic sequences and series. They derive rules for determining the $n$th term of a geometric sequence and the sum of the first $n$ terms of a geometric series. Students solve problems that can be modelled using geometric sequences and series.

The concept of convergence and divergence of infinite geometric sequences and series is introduced through graphing. Students see that the points on some graphs approach a horizontal line while the points on other graphs move away from a horizontal line. They develop an informal understanding about the role of the common ratio in determining convergence and divergence, and derive a rule to determine whether an infinite geometric sequence or series converges. Students learn that the terms of a convergent geometric sequence approach 0 as the term number increases, and they derive a rule to determine the sum of an infinite geometric series that converges.
# Concept Summary

<table>
<thead>
<tr>
<th>Big Ideas</th>
<th>Applying the Big Ideas</th>
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<tr>
<td>• An arithmetic sequence is related to a linear function and is created by repeatedly adding a constant to an initial number. An arithmetic series is the sum of the terms of an arithmetic sequence.</td>
<td>This means that:</td>
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<tr>
<td></td>
<td>• The common difference of an arithmetic sequence is equal to the slope of the line through the points of the graph of the related linear function.</td>
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<td></td>
<td>• Rules can be derived to determine the ( n )th term of an arithmetic sequence and the sum of the first ( n ) terms of an arithmetic series.</td>
</tr>
<tr>
<td>• A geometric sequence is created by repeatedly multiplying an initial number by a constant. A geometric series is the sum of the terms of a geometric sequence.</td>
<td>• The common ratio of a geometric sequence can be determined by dividing any term after the first term by the preceding term.</td>
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<tr>
<td></td>
<td>• Rules can be derived to determine the ( n )th term of a geometric sequence and the sum of the first ( n ) terms of a geometric series.</td>
</tr>
<tr>
<td>• Any finite series has a sum, but an infinite geometric series may or may not have a sum.</td>
<td>• The common ratio determines whether an infinite series has a finite sum.</td>
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</table>
## CURRICULUM OVERVIEW

<table>
<thead>
<tr>
<th>Grade 9</th>
<th>Foundations of Mathematics and Pre-calculus 10</th>
<th>Pre-calculus 11</th>
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<tr>
<td>PR2</td>
<td>Graph linear relations, analyze the graph and interpolate or extrapolate to solve problems.</td>
<td>RF3 Demonstrate an understanding of slope with respect to: rise and run; line segments and lines; rate of change; parallel lines; perpendicular lines.</td>
<td>RF9 Analyze arithmetic sequences and series to solve problems. [CN, PS, R, T]</td>
</tr>
<tr>
<td>PR3</td>
<td>Model and solve problems using linear equations.</td>
<td>RF4 Describe and represent linear relations, using: words; ordered pairs; tables of values; graphs; equations.</td>
<td>RF10 Analyze geometric sequences and series to solve problems. [PS, R, T]</td>
</tr>
<tr>
<td>N1</td>
<td>Demonstrate an understanding of powers with integral bases (excluding base 0) and whole number exponents by: • representing repeated multiplication using powers • using patterns to show that a power with an exponent of zero is equal to one • solving problems involving powers.</td>
<td>RF5 Determine the characteristics of the graphs of linear relations, including the: intercepts; slope; domain; range.</td>
<td>RF9 Graph and analyze exponential and logarithmic functions.</td>
</tr>
</tbody>
</table>

AN3    | Demonstrate an understanding of powers with integral and rational exponents. | RF9 | RF10 Solve problems that involve exponential and logarithmic equations. |
# CHAPTER 1 AT A GLANCE

## Lesson | Timing | Materials and Resources | Program Support
--- | --- | --- | ---
**Chapter Opener**, page 1  
Review prior skills, as relevant. | | | Masters 1.1a, 1.1b  
Activate Prior Learning  
Master 1.6 Chapter Rubric

**1.1 Arithmetic Sequences**, page 2  
Relate linear functions and arithmetic sequences, then solve problems related to arithmetic sequences. | 60 – 75 min | • grid paper  
• scientific calculator  
• graphing calculator (optional)  
Animation | Master 1.1a Activate Prior Learning

**1.2 Arithmetic Series**, page 14  
Derive a rule to determine the sum of \( n \) terms of an arithmetic series, then solve related problems. | 75 min | • scientific calculator  
• graphing calculator (optional)  
Animations | PM 1 Mathematical Dispositions and Learning Skills

**Checkpoint 1**, page 25  
Consolidate content of Lessons 1.1, 1.2. | | • scientific calculator | Master 1.3 Checkpoint 1

**1.3 Geometric Sequences**, page 29  
Solve problems involving geometric sequences. | 75 min | • scientific calculator  
• graphing calculator (optional)  
Animations | Master 1.1b Activate Prior Learning

**1.4 Geometric Series**, page 43  
Derive a rule to determine the sum of \( n \) terms of a geometric series, then solve related problems. | 75 min | • scientific calculator  
• graphing calculator (optional)  
Animation | Master 1.4 Checkpoint 2  
PM 2 Conference Prompts

**1.5 Math Lab: Graphing Geometric Sequences and Series**, page 58  
Investigate the graphs of geometric sequences and geometric series. | 60 – 75 min | • graphing calculator, or computer with graphing software  
• grid paper (optional)  
Dynamic Activity | Master 1.2 Math Lab:  
Graphing Calculator  
Instructions  
PM 3 Observation Record

**1.6 Infinite Geometric Series**, page 63  
Determine the sum of an infinite geometric series. | 75 min | • scientific calculator  
• graphing calculator (optional) | |

**Study Guide, Review, Practice Test**, page 74  
Consolidate and review chapter content, prepare for assessment. | | • scientific calculator | Master 1.5 Chapter Test  
Master 1.7 Chapter Rubric  
Master 1.8 Chapter Summary

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The DVD provides:  
All Program Masters and Chapter 1 masters, as editable and pdf files  
SMART Notebook files for all lessons  
Extra Material for selected lessons
Sequences and Series

BUILDING ON
■ graphing linear functions
■ properties of linear functions
■ expressing powers using exponents
■ solving equations

BIG IDEAS
■ An arithmetic sequence is related to a linear function and is created by repeatedly adding a constant to an initial number. An arithmetic series is the sum of the terms of an arithmetic sequence.

■ A geometric sequence is created by repeatedly multiplying an initial number by a constant. A geometric series is the sum of the terms of a geometric sequence.

■ Any finite series has a sum, but an infinite geometric series may or may not have a sum.

LEADING TO
■ applying the properties of geometric sequences and series to functions that illustrate growth and decay

NEW VOCABULARY

- arithmetic sequence
- term of a sequence or series
- common difference
- infinite arithmetic sequence
- general term
- series
- arithmetic series
- geometric sequence
- common ratio
- finite and infinite geometric sequences
- divergent and convergent sequences
- geometric series
- infinite geometric series
- sum to infinity

TEACHER NOTE
Di: Prerequisite Review
For those students who need it, Master 1.1 provides review (examples and selected exercises) of prior knowledge related to:
a) slopes of the graphs of linear functions;
b) powers and roots

<table>
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<td>Lesson 1.1</td>
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<td>1.1b</td>
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</table>
Focus Relate linear functions and arithmetic sequences, then solve problems related to arithmetic sequences.

Get Started

When the numbers on these plates are arranged in order, the differences between each number and the previous number are the same.

What are the missing numbers?

Let the difference between each pair of numbers be $x$.

Then, $11 + 3x = 35$

$3x = 24$

$x = 8$

The numbers are 19 and 27.

Construct Understanding

Saket took guitar lessons.

The first lesson cost $75 and included the guitar rental for the period of the lessons.

The total cost for 10 lessons was $300.

Suppose the lessons continued.

What would be the total cost of 15 lessons?

Find the cost of one extra lesson.

Cost of 9 lessons $=$ total cost of 10 lessons $-$ total cost of 1st lesson

$= \frac{300}{9} - \frac{75}{9}$

$= \frac{225}{9}$

$= 25$

So, the cost of 1 lesson is: $\frac{225}{9} = 25$

The cost of 15 lessons is: 1 lesson @ $75 + 14$ lessons @ $25$

So, 15 lessons cost: $75 + 14(25) = 425$
In an arithmetic sequence, the difference between consecutive terms is constant. This constant value is called the common difference.

This is an arithmetic sequence:
4, 7, 10, 13, 16, 19, . . .
The first term of this sequence is: \( t_1 = 4 \)
The second term is: \( t_2 = 7 \)

Let \( d \) represent the common difference. For the sequence above:
\[
\begin{align*}
d &= t_2 - t_1 \\
&= 7 - 4 \\
&= 3 \\
d &= t_3 - t_2 \\
&= 10 - 7 \\
&= 3 \\
d &= t_4 - t_3 \\
&= 13 - 10 \\
&= 3
\end{align*}
\]

The dots indicate that the sequence continues forever; it is an infinite arithmetic sequence.

To graph this arithmetic sequence, plot the term value, \( t_n \), against the term number, \( n \).

The graph represents a linear function because the points lie on a straight line. A line through the points on the graph has slope 3, which is the common difference of the sequence.

In an arithmetic sequence, the common difference can be any real number.

Here are some other examples of arithmetic sequences.

- This is an increasing arithmetic sequence because \( d \) is positive and the terms are increasing: \( \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \ldots \); with \( d = \frac{1}{4} \)
- This is a decreasing arithmetic sequence because \( d \) is negative and the terms are decreasing: \( 5, -1, -7, -13, -19, \ldots \); with \( d = -6 \)
Check Your Understanding

1. Write the first 6 terms of:
   a) an increasing arithmetic sequence
   b) a decreasing arithmetic sequence

   a) Sample response: Choose \( t_1 = -20 \) and \( d = 2 \).
   The sequence is: \(-20, -20 + 2, -20 + 4, -20 + 6, -20 + 8, -20 + 10, \ldots\)
   Simplify. An arithmetic sequence is: \(-20, -18, -16, -14, -12, -10, \ldots\)

   b) Sample response: Choose \( t_1 = 100 \) and \( d = -3 \).
   The sequence is: \(100, 100 - 3, 100 - 6, 100 - 9, 100 - 12, 100 - 15, \ldots\)
   Simplify. An arithmetic sequence is: \(100, 97, 94, 91, 88, 85, \ldots\)

Example 1 Writing an Arithmetic Sequence

Write the first 5 terms of:

a) an increasing arithmetic sequence
b) a decreasing arithmetic sequence

SOLUTION

a) Choose any number as the first term; for example, \( t_1 = -7 \).
   The sequence is to increase, so choose a positive common difference; for example, \( d = 2 \). Keep adding the common difference until there are 5 terms.

\[
\begin{align*}
  t_1 & = -7, \\
  t_2 & = -7 + 2, \\
  t_3 & = -7 + 2 + 2, \\
  t_4 & = -7 + 2 + 2 + 2, \\
  t_5 & = -7 + 2 + 2 + 2 + 2
\end{align*}
\]

The arithmetic sequence is: \(-7, -5, -3, -1, 1, \ldots\)

b) Choose the first term; for example, \( t_1 = 5 \).
   The sequence is to decrease, so choose a negative common difference; for example, \( d = -3 \).

\[
\begin{align*}
  t_1 & = 5, \\
  t_2 & = 5 - 3, \\
  t_3 & = 5 - 3 - 3, \\
  t_4 & = 5 - 3 - 3 - 3, \\
  t_5 & = 5 - 3 - 3 - 3 - 3
\end{align*}
\]

The arithmetic sequence is: \(5, 2, -1, -4, -7, \ldots\)

Check Your Understanding 1.

Write the first 6 terms of:

a) an increasing arithmetic sequence
b) a decreasing arithmetic sequence

Answers:

1. a) \(-20, -18, -16, -14, -12, -10, \ldots\)
   b) \(100, 97, 94, 91, 88, 85, \ldots\)
The General Term of an Arithmetic Sequence

An arithmetic sequence with first term, \( t_1 \), and common difference, \( d \), is:

\( t_1, t_1 + d, t_1 + 2d, t_1 + 3d, \ldots \)

The general term of this sequence is:

\[ t_n = t_1 + d(n - 1) \]

### Example 2

Calculating Terms in a Given Arithmetic Sequence

For this arithmetic sequence:

\(-3, 2, 7, 12, \ldots\)

**a)** Determine \( t_{20} \).

**b)** Which term in the sequence has the value 212?

**SOLUTION**

\(-3, 2, 7, 12, \ldots\)

**a)** Calculate the common difference: \( 2 - (-3) = 5 \)

Use: \( t_n = t_1 + d(n - 1) \)

Substitute: \( n = 20, t_1 = -3, d = 5 \)

\[ t_{20} = -3 + 5(20 - 1) \]

Use the order of operations.

\[ t_{20} = -3 + 5(19) \]

\[ t_{20} = 92 \]

**b)** Use: \( t_n = t_1 + d(n - 1) \)

Substitute: \( t_n = 212, t_1 = -3, d = 5 \)

\[ 212 = -3 + 5(n - 1) \]

Solve for \( n \).

\[ 212 = -3 + 5n - 5 \]

\[ 212 = -8 + 5n \]

\[ 220 = 5n \]

\[ 44 = n \]

The term with value 212 is \( t_{44} \).

**THINK FURTHER**

In Example 2, how could you show that 246 is not a term of the sequence?

I would substitute \( t_n = 246, t_1 = -3, \) and \( d = 5 \) in \( t_n = t_1 + d(n - 1) \), and when I solved for \( n \), its value would not be a natural number.
Example 3  Calculating a Term in an Arithmetic Sequence, Given Two Terms

Two terms in an arithmetic sequence are \( t_5 = 4 \) and \( t_8 = 34 \). What is \( t_1 \)?

**SOLUTION**

Let the common difference be \( d \).

From the diagram,

\[
\begin{align*}
&\text{Substitute:} && t_5 = 4 + 3d \\
&\text{Solve for} && 3d = 27 \\
&d && d = 9 \\
&\text{Let the common difference be} && t_i = t_j - 3d \\
&t_i = -4 - 3(9) && t_i = -43 \\
&\text{Then,} && t_1 = \frac{t_8}{5} - 2d \\
&t_1 = 4 - 2(6) && t_1 = 4 - 12 \\
&t_1 = -8 && t_i = -8
\end{align*}
\]

Example 4  Using an Arithmetic Sequence to Model and Solve a Problem

Some comets are called periodic comets because they appear regularly in our solar system. The comet Kojima appears about every 7 years and was last seen in the year 2007. Halley’s comet appears about every 76 years and was last seen in 1986.

Determine whether both comets should appear in 3043.

**SOLUTION**

The years in which each comet appears form an arithmetic sequence.

The arithmetic sequence for Kojima has \( t_1 = 2007 \) and \( d = 7 \).

To determine whether Kojima should appear in 3043, determine whether 3043 is a term of its sequence.

\[
\begin{align*}
&\text{Substitute:} && t_n = 3043, t_1 = 2007, d = 7 \\
&3043 = 2007 + 7(n - 1) && \text{Solve for} \ n. \\
&3043 = 2000 + 7n && 3043 = 2007 + 7n \\
&1043 = 7n && 149 = 7n \\
&149 = n &&
\end{align*}
\]
Since the year 3043 is the 149th term in the sequence, Kojima should appear in 3043.

The arithmetic sequence for Halley’s comet has \( t_1 = 1986 \) and \( d = 76 \).
To determine whether Halley’s comet should appear in 3043, determine whether 3043 is a term of its sequence.

\[
t_n = t_1 + d(n - 1)
\]
Substitute: \( t_n = 3043, t_1 = 1986, d = 76 \)
\[
3043 = 1986 + 76(n - 1)
\]
Solve for \( n \).
\[
3043 = 1910 + 76n
\]
\[
1133 = 76n
\]
\[
n = 14.9078...
\]
Since \( n \) is not a natural number, the year 3043 is not a term in the arithmetic sequence for Halley’s comet; so the comet will not appear in that year.

Discuss the Ideas

1. How can you tell whether a sequence is an arithmetic sequence? What do you need to know to be certain?

   I would calculate the difference between the given consecutive terms. If the differences are equal, the sequence is possibly arithmetic. I can’t be certain unless I know that the sequence continues with the same difference between terms.

2. The definition of an arithmetic sequence relates any term after the first term to the preceding term. Why is it useful to have a rule for determining any term?

   When I use a rule, I don’t have to write all the terms before the term I’m trying to determine. For example, to determine \( t_{50} \), if I didn’t have a rule, I would have to write all the terms from \( t_1 \) to \( t_{50} \).

3. Suppose you know a term of an arithmetic sequence. What information do you need to determine any other term?

   I need to know the position, \( n \), of the given term, \( t_n \), and the common difference, \( d \). Then I can substitute in the expression for \( t_n \) to determine \( t_1 \). Once I know \( t_1 \) and \( d \), I can determine the value of any other term.
Exercises

A

4. Circle each sequence that could be arithmetic. Determine its common difference, \(d\).

a) \(6, 10, 14, 18, \ldots\)
\(d\) is: \(10 - 6 = 4\)

b) \(9, 7, 5, 3, \ldots\)
\(d\) is: \(7 - 9 = -2\)

c) \(-11, -4, 3, 10, \ldots\)
\(d\) is: \(-4 - (-11) = 7\)

Not arithmetic

d) \(2, -4, 8, -16, \ldots\)

5. Each sequence is arithmetic. Determine each common difference, \(d\), then list the next 3 terms.

a) \(12, 15, 18, \ldots\)
\(d\) is: \(15 - 12 = 3\)
The next 3 terms are:
\(18 + 3, 18 + 6, 18 + 9;\)
or \(21, 24, 27\)

b) \(25, 21, 17, \ldots\)
\(d\) is: \(21 - 25 = -4\)
The next 3 terms are:
\(17 - 4, 17 - 8, 17 - 12;\)
or \(13, 9, 5\)

6. Determine the indicated term of each arithmetic sequence.

a) \(6, 11, 16, \ldots; t_7\)
Use: \(t_n = t_1 + d(n - 1)\)
Substitute: \(n = 7, t_1 = 6, d = 5\)
\(t_7 = 6 + 5(7 - 1)\)
\(t_7 = 36\)

b) \(2, 1\frac{1}{2}, 1, \ldots; t_{35}\)
Use: \(t_n = t_1 + d(n - 1)\)
Substitute: \(n = 35, t_1 = 2, d = -\frac{1}{2}\)
\(t_{35} = 2 - \frac{1}{2}(35 - 1)\)
\(t_{35} = -15\)

7. Write the first 4 terms of each arithmetic sequence, given the first term and the common difference.

a) \(t_1 = -3, d = 4\)
\(t_1 = -3\)
\(t_1 + d = -3 + 4, or 1\)
\(t_2 = 1\)
\(t_1 + d = 1 + 4, or 5\)
\(t_3 = 5\)
\(t_1 + d = 5 + 4, or 9\)
\(t_4 = 9\)

b) \(t_1 = -0.5, d = -1.5\)
\(t_1 = -0.5\)
\(t_1 + d = -0.5 - 1.5, or -2\)
\(t_2 = -2\)
\(t_1 + d = -2 - 1.5, or -3.5\)
\(t_3 = -3.5\)
\(t_1 + d = -3.5 - 1.5, or -5\)

B

8. When you know the first term and the common difference of an arithmetic sequence, how can you tell if it is increasing or decreasing? Use examples to explain.

An arithmetic sequence is increasing if \(d\) is positive; for example, when \(t_1 = -10\) and \(d = 3: -10, -7, -4, -1, 2, \ldots\)
An arithmetic sequence is decreasing if \(d\) is negative; for example, when \(t_1 = -10\) and \(d = -3: -10, -13, -16, -19, -22, \ldots\)
9. a) Create your own arithmetic sequence. Write the first 7 terms. Explain your method.

Sample response: I chose $t_1 = 3$ and $d = 5$; I add 5 to 3, then keep adding 5. The first 7 terms of the sequence are: 3, 8, 13, 18, 23, 28, 33

b) Use technology or grid paper to graph the sequence in part a. Plot the Term value on the vertical axis and the Term number on the horizontal axis. Print the graph or sketch it on this grid.

i) How do you know that you have graphed a linear function?

The points lie on a non-vertical straight line.

ii) What does the slope of the line through the points represent? Explain why.

The slope is the common difference because it is the rise when the run is 1; that is, after the first point, each point can be plotted by moving 5 units up and 1 unit right.

10. Two terms of an arithmetic sequence are given. Determine the indicated terms.

a) $t_4 = 24$, $t_{10} = 66$; determine $t_1$

$b = 6d$

Substitute for $t_{10}$ and $t_4$, then solve for $d$.

$66 = 24 + 6d$

$6d = 42$

$d = 7$

$t_1 = t_4 - 3d$

Substitute for $t_4$ and $d$.

$t_1 = 24 - 3(7)$

$t_1 = 3$

b) $t_3 = 81$, $t_{12} = 27$; determine $t_{23}$

$t_{12} = t_3 + 9d$

Substitute for $t_{12}$ and $t_3$.

$27 = 81 + 9d$

$9d = -54$

$d = -6$

$t_{23} = t_{12} + 11d$

Substitute for $t_{12}$ and $d$.

$t_{23} = 27 + 11(-6)$

$t_{23} = -39$

11. Create an arithmetic sequence for each description below. For each sequence, write the first 6 terms and a rule for $t_n$.

a) an increasing sequence

Sample response:

Choose a positive common difference.

Use: $t_1 = 4$ and $d = 3$

The sequence is:

$4, 7, 10, 13, 16, 19, \ldots$

Use: $t_n = t_1 + d(n - 1)$

Substitute: $t_1 = 4, d = 3$

$t_n = 4 + 3(n - 1)$

$t_n = 1 + 3n$

b) a decreasing sequence

Sample response:

Choose a negative common difference.

Use: $t_1 = 4$ and $d = -3$

The sequence is:

$4, 1, -2, -5, -8, -11, \ldots$

Use: $t_n = t_1 + d(n - 1)$

Substitute: $t_1 = 4, d = -3$

$t_n = 4 - 3(n - 1)$

$t_n = 7 - 3n$
12. Claire wrote the first 3 terms of an arithmetic sequence: 3, 6, 9, . . .
When she asked Alex to extend the sequence to the first 10 terms, he wrote:
3, 6, 9, 3, 6, 9, 3, 6, 9, 3, . . .

a) Is Alex correct? Explain.
   
   No, Alex’s sequence is not arithmetic because the terms do not increase or decrease by the same number.

b) What fact did Alex ignore when he extended the sequence?
   
   Alex did not use the common difference of 3 to calculate each term.

c) What is the correct sequence?
   
   Add 3 to get each next term: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, . . .

13. Determine whether 100 is a term of an arithmetic sequence with $t_1 = 250$ and $t_6 = 245.5$.

Let the common difference be $d$.
Use: $t_6 = t_1 + 5d$
Substitute: $t_6 = 245.5, t_1 = 250$
$245.5 = 250 + 3d$
$3d = -4.5$
$d = -1.5$

Use: $t_1 = t_1 - 2d$
Substitute: $t_1 = 250, d = -1.5$
$t_1 = 250 - 2(-1.5)$
$t_1 = 253$

Use: $t_6 = t_1 + 5d$
Substitute: $t_6 = 100, d = -1.5, t_1 = 253$
$100 = 253 - 1.5(n - 1)$
$1.5n = 154.5$
$n = 103$

Since $t_{103} = 100$, then 100 is a term of the sequence.
14. The Chinese zodiac associates years with animals. Ling was born in 1994, the Year of the Dog.
   a) The Year of the Dog repeats every 12 years. List the first three years that Ling will celebrate her birthday in the Year of the Dog.
      Add 12 to 1994 three times: 2006, 2018, 2030
   b) Why do the years in part a form an arithmetic sequence?
      The difference between consecutive dates is constant.
   c) In 2099, Nunavut will celebrate its 100th birthday.
      Will that year also be the Year of the Dog? Explain.
      All terms in the sequence are even numbers; 2099 is an odd number so 2099 cannot be the Year of the Dog.

15. In this arithmetic sequence: 3, 8, 13, 18, . . . ; which term has the value 123?
   \[ d \text{ is: } 8 - 3 = 5 \]
   Use: \( t_n = t_1 + d(n - 1) \)
   Substitute: \( t_n = 123, t_1 = 3, d = 5 \)
   \( 123 = 3 + 5(n - 1) \)
   \( 123 = 3 + 5n - 5 \)
   \( 5n = 125 \)
   \( n = 25 \)
   123 is the 25th term.

16. For two different arithmetic sequences, \( t_5 = -1 \). What are two possible sequences? Explain your reasoning.
   Sample response: For one sequence, choose a number for the common difference, \( d \), such as 5. Keep subtracting 5 to get the preceding terms.
   \( t_1 = -1 \quad t_2 = -1 - 5 \quad t_3 = -6 - 5 \quad t_4 = -11 - 5 \quad t_5 = -16 - 5 \quad \)
   \( = -6 \quad = -11 \quad = -16 \quad = -21 \)
   One arithmetic sequence is: \(-21, -16, -11, -6, -1, . . . \)
   For the other sequence, choose a different number for \( d \), such as \(-8 \). Keep subtracting \(-8 \).
   \( t_1 = -1 \quad t_2 = -1 + 8 \quad t_3 = 7 + 8 \quad t_4 = 15 + 8 \quad t_5 = 23 + 8 \quad \)
   \( = 7 \quad = 15 \quad = 23 \quad = 31 \)
   Another arithmetic sequence is: \(31, 23, 15, 7, -1, . . . \)
17. A sequence is created by adding each term of an arithmetic sequence to the preceding term.

a) Show that the new sequence is arithmetic.

Use the general sequence: \( t_1, t_1 + d, t_1 + 2d, t_1 + 3d, \ldots \)
The new sequence is: \( t_1 + t_1 + d, t_1 + d + t_1 + 2d, t_1 + 2d + t_1 + 3d, \)
\( t_1 + 3d + t_1 + 4d, \ldots \)
This simplifies to: \( 2t_1 + d, 2t_1 + 3d, 2t_1 + 5d, 2t_1 + 7d, \ldots \)
This sequence has first term \( 2t_1 + d \) and common difference \( 2d \), so the sequence is arithmetic.

b) How are the common differences of the two sequences related?

The common difference of the new sequence is double the common difference of the original sequence.

18. In this arithmetic sequence, \( k \) is a natural number: \( k, \frac{2k}{3}, 0, \ldots \)

a) Determine \( t_6 \).

The common difference, \( d \), is: \( \frac{2k}{3} - k = \frac{-k}{3} \)
\( t_1 = 0 \quad t_6 = 0 + \left( \frac{k}{3} \right) \quad t_6 = \frac{-k}{3} - \frac{k}{3} \)
\( = \frac{-2k}{3} \quad = \frac{-2k}{3} \)
\( t_6 \) is \( \frac{-2k}{3} \).

b) Write an expression for \( t_n \).

Use: \( t_n = t_1 + d(n - 1) \) Substitute: \( t_1 = k, d = \frac{-k}{3} \)
\( t_n = k + \left( \frac{-k}{3} \right)(n - 1) \)
\( t_n = k - \frac{kn}{3} + \frac{k}{3} \)
\( t_n = \frac{4k}{3} - \frac{kn}{3} \)

b) Suppose \( t_{20} = -16 \); determine the value of \( k \).

Use: \( t_n = \frac{4k}{3} - \frac{kn}{3} \) Substitute: \( t_n = -16, n = 20 \)
\( -16 = \frac{4k}{3} - \frac{20k}{3} \)
\( -16 = -\frac{16k}{3} \)
\( k = 3 \)
Multiple-Choice Questions

1. How many of these sequences have a common difference of $-4$?
   \[ -19, -15, -11, -7, -3, \ldots \]
   \[ 19, 15, 11, 7, 3, \ldots \]
   \[ 3, 7, 11, 15, 19, \ldots \]
   \[ -3, 7, -11, 15, -19, \ldots \]
   A. 0  B. 1  C. 2  D. 3

2. Which number below is a term of this arithmetic sequence?
   \[ 97, 91, 85, 79, 73, \ldots \]
   A. $-74$  B. $-75$  C. $-76$  D. $-77$

3. The first 6 terms of an arithmetic sequence are plotted on a grid. The coordinates of two points on the graph are (3, 11) and (6, 23). What is an expression for the general term of the sequence?
   A. $6n - 3$  B. $3n + 11$  C. $4n - 1$  D. $1 + 4n$

Study Note

How are arithmetic sequences and linear functions related?

Both an arithmetic sequence and a linear function show constant change. An arithmetic sequence can be described by a linear function whose domain is the natural numbers.

ANSWERS

4. a) 4  b) $-2$  c) 7  5. a) 3, 21, 42, 27  b) $-4; 13, 9, 5$  6. a) 36  b) $-15$
7. a) $-3, 1, 5, 9$  b) $-0.5, -2, -3.5, -5$  10. a) 3  b) $-39$
12. a) no  c) 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, $\ldots$  13. 100 is a term.
14. a) 2006, 2018, 2030  c) no  15. $t_3$  18. a) $\frac{-2k}{3}$  b) $\frac{4k}{3} - \frac{k}{3}$  c) 3

Multiple Choice

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1.2 Arithmetic Series

**FOCUS** Derive a rule to determine the sum of *n* terms of an arithmetic series, then solve related problems.

**Get Started**
Suppose this sequence continues. What is the value of the 8th term?

- The 7th term is 13, the common difference is 2, so the 8th term is 15.

What is an expression for the *n*th term?

- Each term value is 1 less than 2 times the term number, so the *n*th term is $2n - 1$.

Is 50 a term in this sequence? How do you know?

- No, because 50 is an even number and all the term values are odd numbers.

**Construct Understanding**
Talise displayed 90 photos of the Regina Dragon Boat Festival in 5 rows. The difference between the numbers of photos in consecutive rows was constant.

How many different sequences are possible? Justify your answer.

- Since the differences in the numbers of photos in consecutive rows are constant, these numbers form an arithmetic sequence.

  Suppose the common difference is 0. Then the rows would have the same numbers of photos, so the number of photos in each row would be: $\frac{90}{5} = 18$

  Suppose the common difference is 1. Start with 18 photos in the 3rd (middle) row. Then, the 4th row has $18 + 1 = 19$; and the 5th row has $19 + 1 = 20$.

  The 2nd row has $18 - 1 = 17$; and the 1st row has $17 - 1 = 16$
For each possible sequence, there are 18 photos in the 3rd row.
So, add and subtract possible common differences to get the numbers of photos in the other rows.

<table>
<thead>
<tr>
<th>Difference</th>
<th>Number of Photos in Rows 1 to 5</th>
<th>Difference</th>
<th>Number of Photos in Rows 1 to 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>−8</td>
<td>34, 26, 18, 10, 2</td>
<td>1</td>
<td>16, 17, 18, 19, 20</td>
</tr>
<tr>
<td>−7</td>
<td>32, 25, 18, 11, 4</td>
<td>2</td>
<td>14, 16, 18, 20, 22</td>
</tr>
<tr>
<td>−6</td>
<td>30, 24, 18, 12, 6</td>
<td>3</td>
<td>12, 15, 18, 21, 24</td>
</tr>
<tr>
<td>−5</td>
<td>28, 23, 18, 13, 8</td>
<td>4</td>
<td>10, 14, 18, 22, 26</td>
</tr>
<tr>
<td>−4</td>
<td>26, 22, 18, 14, 10</td>
<td>5</td>
<td>8, 13, 18, 23, 28</td>
</tr>
<tr>
<td>−3</td>
<td>24, 21, 18, 15, 12</td>
<td>6</td>
<td>6, 12, 18, 24, 30</td>
</tr>
<tr>
<td>−2</td>
<td>22, 20, 18, 16, 14</td>
<td>7</td>
<td>4, 11, 18, 25, 32</td>
</tr>
<tr>
<td>−1</td>
<td>20, 19, 18, 17, 16</td>
<td>8</td>
<td>2, 10, 18, 26, 34</td>
</tr>
<tr>
<td>0</td>
<td>18, 18, 18, 18, 18</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

So, 17 different sequences are possible.

A series is a sum of the terms in a sequence.
An arithmetic series is the sum of the terms in an arithmetic sequence.

For example, an arithmetic sequence is: 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50, 53, 56.
The related arithmetic series is: 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 + 38 + 41 + 44 + 47 + 50 + 53 + 56.
The term, $S_n$, is used to represent the sum of the first $n$ terms of a series. The $n$th term of an arithmetic series is the $n$th term of the related arithmetic sequence.

For the arithmetic series above:

- $S_1 = t_1$
- $S_2 = t_1 + t_2$
- $S_3 = t_1 + t_2 + t_3$
- $S_4 = t_1 + t_2 + t_3 + t_4$
- $S_5 = 5$
- $S_6 = 5 + 8$
- $S_7 = 5 + 8 + 11$
- $S_8 = 5 + 8 + 11 + 14$
- $S_9 = 13$
- $S_{10} = 24$
- $S_{11} = 38$

These are called partial sums.

If there are only a few terms, $S_n$ can be determined using mental math.
To develop a rule to determine $S_n$, use algebra.
Write the sum on one line, reverse the order of the terms on the next line, then add vertically. Write the sum as a product.
Chapter 1: Sequences and Series

THINK FURTHER

Why can the \((n - 1)\)th term be written as \(t_n - d\)?

The \((n - 1)\)th term comes immediately before the \(n\)th term. So, the value of the \((n - 1)\)th term is the value of the \(n\)th term minus the common difference.

\[
S_n = t_1 + (t_1 + d) + (t_1 + 2d) + \ldots + (t_n - d) + t_n
\]
\[
S_n = t_n + (t_n - d) + (t_n - 2d) + \ldots + (t_1 + d) + t_1
\]
\[
2S_n = (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \ldots + (t_1 + t_n) + (t_1 + t_n)
\]
\[
2S_n = n(t_1 + t_n)
\]
\[
S_n = \frac{n(t_1 + t_n)}{2}
\]

The rule above is used when \(t_1\) and \(t_n\) are known. Substitute for \(t_n\) to write \(S_n\) in a different way, so it can be used when \(t_1\) and the common difference, \(d\), are known.

\[
S_n = \frac{n(t_1 + t_n)}{2}
\]
Substitute: \(t_n = t_1 + d(n - 1)\)

\[
S_n = \frac{n(t_1 + t_1 + d(n - 1))}{2}
\]
Combine like terms.

\[
S_n = \frac{n[2t_1 + d(n - 1)]}{2}
\]

The Sum of \(n\) Terms of an Arithmetic Series

For an arithmetic series with 1st term, \(t_1\), common difference, \(d\), and \(n\)th term, \(t_n\), the sum of the first \(n\) terms, \(S_n\), is:

\[
S_n = \frac{n(t_1 + t_n)}{2}
\]

or

\[
S_n = \frac{n[2t_1 + d(n - 1)]}{2}
\]

Example 1 Determining the Sum, Given the Series

Determine the sum of the first 6 terms of this arithmetic series:

\[-75 - 69 - 63 - 57 - 51 - 45 \ldots\]

**SOLUTION**

\[-75 - 69 - 63 - 57 - 51 - 45 \ldots\]
\(t_1\) is \(-75\) and \(t_6\) is \(-45\).

Use: \(S_n = \frac{n(t_1 + t_n)}{2}\) Substitute: \(n = 6, t_1 = -75, t_n = -45\)

\[
S_6 = \frac{6(-75 - 45)}{2}
\]

\[
S_6 = -360
\]

The sum of the first 6 terms is \(-360\).
Example 2  Determining the Sum, Given the First Term and Common Difference

An arithmetic series has \( t_1 = 5.5 \) and \( d = -2.5 \); determine \( S_{40} \).

**SOLUTION**

Use: \( S_n = \frac{n[2t_1 + d(n - 1)]}{2} \)

Substitute: \( n = 40, t_1 = 5.5, d = -2.5 \)

\[
S_{40} = \frac{40[2(5.5) - 2.5(40 - 1)]}{2}
\]

\[
S_{40} = -1730
\]

Example 3  Determining the First Few Terms Given the Sum, Common Difference, and One Term

An arithmetic series has \( S_{20} = 143\frac{1}{3} \), \( d = \frac{1}{3} \), and \( t_{20} = 10\frac{1}{3} \); determine the first 3 terms of the series.

**SOLUTION**

\( S_{20} \) and \( t_{20} \) are known, so use this rule to determine \( t_1 \):

\[
S_n = \frac{n(t_1 + t_n)}{2}
\]

Substitute: \( n = 20, S_{20} = 143\frac{1}{3}, t_{20} = 10\frac{1}{3} \)

\[
143\frac{1}{3} = \frac{20(t_1 + 10\frac{1}{3})}{2}
\]

Simplify.

\[
143\frac{1}{3} = 10(t_1 + 10\frac{1}{3})
\]

\[
143\frac{1}{3} = 10t_1 + 103\frac{1}{3}
\]

Solve for \( t_1 \).

\[
40 = 10t_1
\]

\[
4 = t_1
\]

The first term is 4 and the common difference is \( \frac{1}{3} \).

So, the first 3 terms of the series are written as the partial sum:

\[
4 + 4\frac{1}{3} + 4\frac{2}{3}
\]

THINK FURTHER

In Example 3, which partial sums are natural numbers? Why?

\( S_n \) is a natural number, and \( t_1 + t_2 \) is a natural number, so \( S_2 = S_1 + t_1 + t_2 \) is a natural number. Since \( t_1 \) is a natural number, then \( S_2 = S_1 + t_2 \) is a natural number. Since \( t_1 + t_2 \) is a natural number, then \( S_3 = S_1 + t_3 + t_2 \) is a natural number. This pattern continues. The partial sums that are natural numbers are: \( S_1, S_3, S_4, S_6, S_8, S_{10}, S_{15} \), and so on.

Check Your Understanding

2. An arithmetic series has \( t_1 = 3 \) and \( d = -4 \); determine \( S_{35} \).

Use: \( S_n = \frac{n[2t_1 + d(n - 1)]}{2} \)

Substitute: \( n = 25, t_1 = 3, d = -4 \)

\[
S_{35} = \frac{25[2(3) - 4(25 - 1)]}{2}
\]

\[
S_{35} = -1125
\]

3. An arithmetic series has \( S_{15} = 93.75, d = 0.75, \) and \( t_{15} = 11.5 \); determine the first 3 terms of the series.

\( S_{15} \) and \( t_{15} \) are known, so use this rule:

\[
S_n = \frac{n(t_1 + t_n)}{2}
\]

Substitute: \( n = 15, S_{15} = 93.75, t_{15} = 11.5 \)

93.75 = \frac{15(t_1 + 11.5)}{2}

93.75 = 7.5(t_1 + 11.5)

93.75 = 7.5t_1 + 86.25

7.5 = 7.5t_1

1 = t_1

The 1st term is 1.

The 2nd term is: \( 1 + 0.75 = 1.75 \)

The 3rd term is: \( 1.75 + 0.75 = 2.5 \)

So, the first 3 terms of the series are: \( 1, 1.75, 2.5 \)

Check Your Understanding

Answers:

2. \(-1125\)

3. \(1 + 1.75 + 2.5\)
Example 4  Using an Arithmetic Series to Model and Solve a Problem

Students created a trapezoid from the cans they had collected for the food bank. There were 10 rows in the trapezoid. The bottom row had 100 cans. Each consecutive row had 5 fewer cans than the previous row. How many cans were in the trapezoid?

SOLUTION

The numbers of cans in the rows form an arithmetic sequence with first 3 terms 100, 95, 90, . . .

The total number of cans is the sum of the first 10 terms of the arithmetic series:

\[ S_{10} = \frac{n}{2} (t_1 + t_n) \]

Use: Substitute:

\[ n = 10, \ t_1 = 100, \ d = -5 \]

\[ S_{10} = \frac{10}{2}(2(100) - 5(10 - 1)) \]

\[ S_{10} = 775 \]

There were 775 cans in the trapezoid.

Check Your Understanding

4. The bottom row in a trapezoid had 49 cans. Each consecutive row had 4 fewer cans than the previous row. There were 11 rows in the trapezoid. How many cans were in the trapezoid?

The numbers of cans in the rows form an arithmetic sequence with first 3 terms 49, 45, 41, . . .

Use: \[ S_n = \frac{n}{2} [2t_1 + d(n - 1)] \]
Substitute: \[ n = 11, \ t_1 = 49, \ d = -4 \]

\[ S_{11} = \frac{11}{2} [2(49) - 4(11 - 1)] \]

\[ S_{11} = 319 \]

There were 319 cans in the trapezoid.

TEACHER NOTE

DI: Common Difficulties

Some students may have difficulty recalling the sum formula that involves the first term and common difference of an arithmetic series, as applied in Example 2. They could use the given common difference to determine the nth term of the series, then use the formula:

\[ S_n = \frac{n(t_1 + t_n)}{2} \]

Discuss the Ideas

1. How are an arithmetic series and an arithmetic sequence related?

The terms of an arithmetic series form an arithmetic sequence. The terms of an arithmetic sequence are added to form an arithmetic series.

2. Suppose you know the 1st and nth terms of an arithmetic series. What other information do you need to determine the value of n?

When I know \( t_1 \) and \( t_n \), I need to know the common difference, \( d \), then I can use the rule \( t_n = t_1 + d(n - 1) \) to determine \( n \). Or, when I know \( t_1 \) and \( t_n \), I need to know the sum of the first \( n \) terms, \( S_n \), then I can use the rule \( S_n = \frac{n(t_1 + t_n)}{2} \) to determine \( n \).
Exercises

A

3. Use each arithmetic sequence to write the first 4 terms of an arithmetic series.
   a) 2, 4, 6, 8, . . .
      \[2 + 4 + 6 + 8\]
   b) -2, 3, 8, 13, . . .
      \[-2 + 3 + 8 + 13\]
   c) 4, 0, -4, -8, . . .
      \[4 + 0 - 4 - 8\]
   d) \[\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}, . . .\]
      \[\frac{1}{2} + \frac{1}{4} + 0 - \frac{1}{4}\]

4. Determine the sum of the given terms of each arithmetic series.
   a) \[12 + 10 + 8 + 6 + 4\]
      \[12 + 10 + 8 + 6 + 4 = 40\]
   b) \[-2 - 4 - 6 - 8 - 10\]
      \[-2 - 4 - 6 - 8 - 10 = -30\]

5. Determine the sum of the first 20 terms of each arithmetic series.
   a) \[3 + 7 + 11 + 15 + \ldots\]
      Use: \[S_{20} = \frac{n(2t_1 + d(n - 1))}{2}\]
      Substitute:
      \[n = 20, t_1 = 3, d = 4\]
      \[S_{20} = \frac{20[2(3) + 4(20 - 1)]}{2}\]
      \[S_{20} = 820\]
   b) \[-21 - 15.5 - 10 - 4.5 - \ldots\]
      Use: \[S_{20} = \frac{n(2t_1 + d(n - 1))}{2}\]
      Substitute:
      \[n = 20, t_1 = -21, d = 5.5\]
      \[S_{20} = \frac{20[2(-21) + 5.5(20 - 1)]}{2}\]
      \[S_{20} = 625\]

B

6. For each arithmetic series, determine the indicated value.
   a) \[-4 - 11 - 18 - 25 - \ldots; \text{ determine } S_{28}\]
      Use: \[S_{28} = \frac{n(2t_1 + d(n - 1))}{2}\]
      Substitute:
      \[n = 28, t_1 = -4, d = -7\]
      \[S_{28} = \frac{28[2(-4) - 7(28 - 1)]}{2}\]
      \[S_{28} = -2758\]
   b) \[1 + 3.5 + 6 + 8.5 + \ldots; \text{ determine } S_{42}\]
      Use: \[S_{42} = \frac{n(2t_1 + d(n - 1))}{2}\]
      Substitute:
      \[n = 42, t_1 = 1, d = 2.5\]
      \[S_{42} = \frac{42[2(1) + 2.5(42 - 1)]}{2}\]
      \[S_{42} = 2194.5\]
7. Use the given data about each arithmetic series to determine the indicated value.

a) \( S_{20} = -850 \) and \( t_{20} = -90 \); determine \( t_1 \)

Use: \( S_n = \frac{n(t_1 + t_n)}{2} \)

Substitute:
\[
S_n = -850, \ n = 20, \ t_n = -90
\]
\[
-850 = \frac{20(t_1 - 90)}{2}
\]
\[
-850 = 10t_1 - 1800
\]
\[
t_1 = 5
\]

b) \( S_{15} = 322.5 \) and \( t_1 = 4 \); determine \( d \)

Use: \( S_n = \frac{n(t_1 + d(n - 1))}{2} \)

Substitute:
\[
S_n = 322.5, \ n = 15, \ t_1 = 4
\]
\[
322.5 = \frac{15[2(4) + d(15 - 1)]}{2}
\]
\[
322.5 = 7.5(8 + 14d)
\]
\[
43 = 8 + 14d
\]
\[
d = 2.5
\]

c) \( S_n = -126, \ t_1 = -1, \) and \( t_n = -20 \); determine \( n \)

Use: \( S_n = \frac{n(t_1 + t_n)}{2} \)

Substitute:
\[
S_n = -126, \ t_1 = -1, \ t_n = -20
\]
\[
-126 = \frac{n(-1 - 20)}{2}
\]
\[
-252 = -21n
\]
\[
n = 12
\]

d) \( t_1 = 1.5 \) and \( t_{20} = 58.5 \); determine \( S_{15} \)

Use \( t_n = t_1 + d(n - 1) \) to determine \( d \).
Substitute: \( t_n = 58.5, \ t_1 = 1.5, \ n = 20 \)
\[
58.5 = 1.5 + d(20 - 1)
\]
\[
57 = 19d
\]
\[
d = 3
\]
Use: \( S_n = \frac{n(t_1 + d(n - 1))}{2} \)

Substitute: \( n = 15, \ t_1 = 1.5, \ d = 3 \)
\[
S_{15} = \frac{15[2(1.5) + 3(15 - 1)]}{2}
\]
\[
S_{15} = \frac{15(3 + 42)}{2}
\]
\[
S_{15} = 337.5
\]

8. Two hundred seventy-six students went to a powwow. The first bus had 24 students. The numbers of students on the buses formed an arithmetic sequence. What additional information do you need to determine the number of buses? Explain your reasoning.

I need to know the number of students on the last bus, then I can use the rule \( S_n = \frac{n(t_1 + t_n)}{2} \). Or, I need to know the common difference, \( d \), then I can use the rule \( S_n = \frac{n(2t_1 + d(n - 1))}{2} \). In each rule, I substitute what I know, then solve for \( n \).
9. Ryan’s grandparents loaned him the money to purchase a BMX bike. He agreed to repay $25 at the end of the first month, $30 at the end of the second month, $35 at the end of the third month, and so on. Ryan repaid the loan in 12 months. How much did the bike cost? How do you know your answer is correct?

Ryan’s repayments form an arithmetic series with 12 terms, where the 1st term is his first payment, and the common difference is $5.

Use: $S_n = \frac{n(t_1 + t_n)}{2}$ Substitute: $n = 12, t_1 = 25, d = 5$

$S_{12} = \frac{12[2(25) + 5(12 - 1)]}{2}$

$S_{12} = 6(50 + 55)$

$S_{12} = 630$

The bike cost $630.

I used a calculator to add the 12 payments to check that the answer is the same.

10. Determine the sum of the indicated terms of each arithmetic series.

a) $31 + 35 + 39 + \ldots + 107$

Use $t_n = t_1 + d(n - 1)$ to determine $n$. Substitute:

$t_n = 107, t_1 = 31, and d = 4$

$107 = 31 + 4(n - 1)$

$76 = 4n - 4$

$4n = 80$

$n = 20$

Use: $S_n = \frac{n(t_1 + t_n)}{2}$ Substitute:

$n = 20, t_1 = 31, and t_n = 107$

$S_{20} = \frac{20(31 + 107)}{2}$

$S_{20} = 1380$

b) $-13 - 10 - 7 - \ldots + 62$

Use $t_n = t_1 + d(n - 1)$ to determine $n$. Substitute:

$t_n = 62, t_1 = -13, and d = 3$

$62 = -13 + 3(n - 1)$

$75 = 3n - 3$

$3n = 78$

$n = 26$

Use: $S_n = \frac{n(t_1 + t_n)}{2}$ Substitute:

$n = 26, t_1 = -13, and t_n = 62$

$S_{26} = \frac{26(-13 + 62)}{2}$

$S_{26} = 637$

11. a) Explain how this series could be arithmetic.

$1 + 3 + \ldots$

This series could be arithmetic if each term was calculated by adding 2 to the preceding term.

b) What information do you need to be certain that this is an arithmetic series?

I need to know that the number added each time is 2.
12. An arithmetic series has $S_{10} = 100$, $t_1 = 1$, and $d = 2$. How can you use this information to determine $S_{11}$ without using a rule for the sum of an arithmetic series? What is $S_{11}$?

First, we find $t_{11}$ using the formula:

$$t_n = t_1 + (n - 1)d$$

Substituting $n = 11$, $t_1 = 1$, and $d = 2$:

$$t_{11} = 1 + 2(11 - 1) = 21$$

Then $S_{11} = S_{10} + t_{11}$.

The sum of the side lengths form an arithmetic sequence with 4 terms, where $t_4 = 29$ and $S_4 = 74$.

Use $S_n = \frac{n(t_1 + t_n)}{2}$ to determine $t_1$.

Substitute: $S_n = 74$, $n = 4$, $t_n = 29$

$$74 = \frac{4(t_1 + 29)}{2}$$

$$74 = 2t_1 + 58$$

$$16 = 2t_1$$

$$t_1 = 8$$

Use $t_n = t_1 + d(n - 1)$ to determine $d$.

Substitute: $t_n = 29$, $t_1 = 8$, $n = 4$

$$29 = 8 + d(4 - 1)$$

$$21 = 3d$$

$$d = 7$$

So, the other side lengths are: 8 cm, 15 cm, and 22 cm.

13. The side lengths of a quadrilateral form an arithmetic sequence. The perimeter is 74 cm. The longest side is 29 cm. What are the other side lengths?

The sum of the side lengths form an arithmetic series with 4 terms, where $t_4 = 29$ and $S_4 = 74$.

Use $S_n = \frac{n(t_1 + t_n)}{2}$ to determine $t_1$.

Substitute: $S_n = 74$, $n = 4$, $t_n = 29$

$$74 = \frac{4(t_1 + 29)}{2}$$

$$74 = 2t_1 + 58$$

$$16 = 2t_1$$

$$t_1 = 8$$

Use $t_n = t_1 + d(n - 1)$ to determine $d$.

Substitute: $t_n = 29$, $t_1 = 8$, $n = 4$

$$29 = 8 + d(4 - 1)$$

$$21 = 3d$$

$$d = 7$$

So, the other side lengths are: 8 cm, 15 cm, and 22 cm.

14. Derive a rule for the sum of the first $n$ natural numbers:

$$1 + 2 + 3 + \ldots + n$$

The sum of the numbers is an arithmetic series with $t_1 = 1$, $d = 1$, and $t_n = n$.

Use: $S_n = \frac{n(2t_1 + d(n - 1))}{2}$

Substitute: $t_1 = 1$, $d = 1$

$$S_n = \frac{n(2(1) + 1(n - 1))}{2}$$

$$S_n = \frac{n(n + 1)}{2}$$
15. The sum of the first 5 terms of an arithmetic series is 170. The sum of the first 6 terms is 225. The common difference is 7. Determine the first 4 terms of the series.

\[ S_5 = 170, S_6 = 225 \]
\[ t_6 = S_6 - S_5 = 225 - 170 = 55 \]

Use \( t_n = t_1 + d(n - 1) \) to determine \( t_n \).

Substitute: \( t_n = 55, d = 7, n = 6 \)
\[ t_6 = t_1 + 7(6 - 1) \]
\[ t_1 = 20 \]

So, \( t_2 = 27, t_3 = 34, \) and \( t_4 = 41 \)

The first 4 terms are: 20, 27, 34, and 41.

16. The sum of the first \( n \) terms of an arithmetic series is: \( S_n = 3n^2 - 8n \)

Determine the first 4 terms of the series.

Determine \( S_1, S_2, S_3, \) and \( S_4 \).

In \( S_n = 3n^2 - 8n \):

Substitute: \( n = 1 \) \hspace{1cm} Substitue: \( n = 2 \)
\[ S_1 = 3(1)^2 - 8(1) \hspace{1cm} S_2 = 3(2)^2 - 8(2) \]
\[ = -5 \hspace{1cm} = -4 \]

Substitute: \( n = 3 \) \hspace{1cm} Substitute: \( n = 4 \)
\[ S_3 = 3(3)^2 - 8(3) \hspace{1cm} S_4 = 3(4)^2 - 8(4) \]
\[ = 3 \hspace{1cm} = 16 \]
\[ t_1 = S_1 \hspace{1cm} t_1 = S_2 - S_1 \hspace{1cm} t_1 = S_3 - S_2 \hspace{1cm} t_4 = S_4 - S_3 \]
\[ = -5 \hspace{1cm} = 1 \hspace{1cm} = 7 \hspace{1cm} = 13 \]

17. Each number from 1 to 60 is written on one of 60 index cards. The cards are arranged in rows with equal lengths, and no cards are left over. The sum of the numbers in each row is 305. How many rows are there?

Determine the sum of the first 60 natural numbers:
\[ 1 + 2 + 3 + \ldots + 60 \]

This is an arithmetic series with \( t_1 = 1, d = 1, \) and \( t_{60} = 60 \).

Use: \( S_n = \frac{n(t_1 + t_n)}{2} \)

Substitute: \( n = 60, t_1 = 1, t_n = 60 \)
\[ S_n = \frac{60(1 + 60)}{2} \]
\[ S_n = 1830 \]

The sum of the numbers in each row is 305, so the number of rows is:
\[ 1830 = 6 \]
18. Determine the arithmetic series that has these partial sums: $S_4 = 26$, $S_5 = 40$, and $S_6 = 57$

$$
t_6 = S_6 - S_5 = 57 - 40 = 17 \\
t_5 = S_5 - S_4 = 40 - 26 = 14 \\
d = t_6 - t_5 = 3 \\
t_4 = t_3 + d = 11 \\
t_3 = t_2 + d = 8 \\
t_2 = t_1 + d = 5 \\
t_1 = t_2 = -3 \\

The arithmetic series is: $2 + 5 + 8 + 11 + 14 + 17 + \ldots$

### Multiple-Choice Questions

1. Which of these series could be arithmetic?
   - A. $2.5 + 5 + 7.5 + 11 + \ldots$
   - B. $-2.5 - 5 - 7.5 - 11 - \ldots$
   - C. $3.5 + 6 + 8.5 + 11 + \ldots$
   - D. $3.5 - 6 - 8.5 - 11 + \ldots$

2. For which arithmetic series below is 115 the sum of the first 10 terms?
   - A. $39 + 34 + 29 + 24 + 19 + \ldots$
   - B. $-11 - 6 - 1 + 4 + 9 + \ldots$
   - C. $11 + 6 + 1 - 4 - 9 - \ldots$
   - D. $39 + 31 + 23 + 15 + 7 + \ldots$

3. How many of these expressions could be used to determine the sum to $n$ terms of an arithmetic series?

$$
\frac{n(t_1 + t_n)}{2} \quad \frac{n(t_1 + d(n - 1))}{2} \quad \frac{n(t_1 + t_n)}{2} \quad \frac{n(t_1 - t_n)}{2}
$$

   - A. 4
   - B. 3
   - C. 2
   - D. 1

### Study Note

There are two forms of the rule to determine the sum of the first $n$ terms of an arithmetic series. When would you use each form of the rule?

I would use $S_n = \frac{n(t_1 + t_n)}{2}$ when I know the number of terms, the first term, and the common difference. I would use $S_n = \frac{n(t_1 + t_n)}{2}$ when I know the number of terms, the first term, and the $n$th term.

### ANSWERS

3. a) $2 + 4 + 6 + 8$  
   b) $-2 + 3 + 8 + 13$  
   c) $4 + 0 - 4 - 8$  
   d) $\frac{1}{2} + \frac{1}{2} + 0 - \frac{1}{2}  
4. a) 40  
   b) 40  
   c) 30  
   d) 820  
   e) 625  
6. a) $-2758$

b) $2194.5$  
7. a) 5  
   b) 2.5  
   c) 12  
   d) 337.5  
9. $5630$  
10. a) $1380$  
   b) $637$

12. 121  
13. 8 cm, 15 cm, and 22 cm  
14. $S_n = \frac{n(n + 1)}{2}$  
15. 20 + 27 + 34 + 41  
16. $-5 + 1 + 7 + 13$

17. 6  
18. 2 + 5 + 8 + 11 + 14 + 17 + \ldots

### Multiple Choice

1. C  
   2. B  
   3. D
### CHECKPOINT 1

**Self-Assess**

<table>
<thead>
<tr>
<th>Can you . . .</th>
<th>To check, try question . . .</th>
<th>For review, see . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>write an example of an arithmetic sequence and explain how you know it is arithmetic?</td>
<td>2</td>
<td>Page 4 in Lesson 1.1</td>
</tr>
<tr>
<td>explain the meaning of the symbols n, t₁, tₙ, and d?</td>
<td></td>
<td>Page 3 in Lesson 1.1</td>
</tr>
<tr>
<td>identify the assumptions made when defining an arithmetic sequence or series?</td>
<td></td>
<td>Page 7 in Lesson 1.1</td>
</tr>
<tr>
<td>use a rule to determine the nth term in an arithmetic sequence?</td>
<td>3</td>
<td>Page 4 in Lesson 1.1</td>
</tr>
<tr>
<td>describe the relationship between an arithmetic sequence and a linear function?</td>
<td></td>
<td>Page 3 in Lesson 1.1</td>
</tr>
<tr>
<td>use a rule to determine t₁ and n in an arithmetic sequence given the values of t₁ and d?</td>
<td>3b, 4b</td>
<td>Page 5 in Lesson 1.1</td>
</tr>
<tr>
<td>use a rule to determine t₁ and d in an arithmetic sequence given the values of tₙ and n?</td>
<td>4a</td>
<td>Page 6 in Lesson 1.1</td>
</tr>
<tr>
<td>solve problems involving arithmetic sequences?</td>
<td>5</td>
<td>Pages 6-7 in Lesson 1.1</td>
</tr>
<tr>
<td>use a rule to determine the sum Sₙ of an arithmetic series?</td>
<td></td>
<td>Page 16 in Lesson 1.2</td>
</tr>
<tr>
<td>use a rule to determine Sₙ in an arithmetic series given the values of n, t₁, and tₙ?</td>
<td>7b, 8</td>
<td>Page 16 in Lesson 1.2</td>
</tr>
<tr>
<td>use a rule to determine Sₙ in an arithmetic series given the values of n, tₙ, and d?</td>
<td></td>
<td>Page 17 in Lesson 1.2</td>
</tr>
<tr>
<td>use a rule to determine t₁ in an arithmetic series given the values of Sₙ, tₙ, and d?</td>
<td>9a</td>
<td>Page 17 in Lesson 1.2</td>
</tr>
<tr>
<td>solve problems involving arithmetic series?</td>
<td></td>
<td>Page 18 in Lesson 1.2</td>
</tr>
</tbody>
</table>

**TEACHER NOTE**

Have students complete the mind map on Master 1.3a to summarize their knowledge of arithmetic series.
Assess Your Understanding

1. **Multiple Choice** Which arithmetic sequence has \( d = -8 \) and \( t_{10} = -45? \)
   - A. 27, 19, 11, 3, . . .
   - B. -8, -12, -16, -20, . . .
   - C. -5, -13, -21, -29, . . .
   - D. -27, -19, -11, -3, . . .

2. Write the first 4 terms of an arithmetic sequence with its 5th term equal to -4.

   Sample response: I chose a common difference of 2.
   \[ t_1 = -4; \text{ so } t_2 = -4 - 2 = -6; \text{ } t_3 = -6 - 2 = -8; \text{ and } t_5 = -10 - 2 = -12 \]
   My arithmetic sequence is: -12, -10, -8, -6, . . .

3. This sequence is arithmetic: -8, -11, -14, . . .
   a) Write a rule for the \( n \)th term.
      Use: \( t_n = t_1 + d(n - 1) \)
      Substitute: \( t_1 = -8, d = -3 \)
      \( t_n = -8 - 3(n - 1) \)
      \( t_n = -5 - 3n \)
   b) Use your rule to determine the 17th term.
      For \( t_{17} \), use \( t_n = -5 - 3n \) and substitute: \( n = 17 \)
      \( t_{17} = -5 - 3(17) \)
      \( t_{17} = -56 \)

4. Use the given data about each arithmetic sequence to determine the indicated values.
   a) \( t_4 = -5 \) and \( t_7 = -20 \); determine \( d \) and \( t_i \)
      Use: \( t_i = t_1 + 3d \)
      Substitute: \( t_7 = -20, t_4 = -5 \)
      \(-20 = -5 + 3d \)
      \(-15 = 3d \)
      \( d = -5 \)
      Use: \( t_i = t_4 - 3d \)
      Substitute: \( t_4 = -5, d = -5 \)
      \( t_i = -5 - 3(-5) \)
      \( t_i = 10 \)
b) \( t_1 = 3, d = 4, \) and \( t_n = 59; \) determine \( n \)

Use: \( t_n = t_1 + d(n - 1) \) Substitute: \( t_n = 59, t_1 = 3, d = 4 \)

\[
59 = 3 + 4(n - 1) \\
56 = 4n - 4 \\
4n = 60 \\
n = 15
\]

5. The steam clock in the Gastown district of Vancouver, B.C., displays the time on four faces and announces the quarter hours with a whistle chime that plays the tune Westminster Quarters. This sequence represents the number of tunes played from 1 to 3 days: 96, 192, 288, \ldots \) Determine the number of tunes played in one year.

In one year, there are 365 days and 96(365), or 35,040 quarters. So, in one year, 35,040 tunes are played.

1.2

6. **Multiple Choice** For which series could you use \( S_n = \frac{n(t_1 + t_n)}{2} \) to determine its sum?

A. \( 3 + 5 + 7 + 10 + 13 + 17 + 21 \)
B. \( -3 + 1 - 5 - 9 - 13 - 17 - 21 \)
C. \( -3 - 5 - 8 - 10 - 13 - 15 - 18 \)
D. \( 3 - 1 + 5 - 3 + 7 - 5 + 9 \)

7. a) Create the first 5 terms of an arithmetic series with a common difference of \(-3\).

Sample response: I chose a first term of 7.

\( t_1 = 7; \) so \( t_2 \) is \( 7 - 3 = 4; \) \( t_2 \) is \( 4 - 3 = 1; \) \( t_3 \) is \( 1 - 3 = -2; \) and \( t_4 \) is \(-2 - 3 = -5\);

My arithmetic series is: \( 7 + 4 + 1 - 5 - \ldots \)

b) Determine \( S_{26} \) for your series.

Sample response:

Use: \( S_n = \frac{n[2t_1 + d(n - 1)]}{2} \) Substitute: \( n = 26, t_1 = 7, d = -3 \)

\[
S_{26} = \frac{26[2(7) + 3(26 - 1)]}{2} \\
S_{26} = -793
\]
8. Determine the sum of this arithmetic series:
\[-2 + 3 + 8 + 13 + \ldots + 158\]
To determine \( n \), use \( t_n = t_1 + (n - 1)d \)
Substitute: \( t_n = 158, t_1 = -2, d = 5 \)
\[158 = -2 + 5(n - 1)\]
\[160 = 5n - 5\]
\[165 = 5n\]
\[n = 33\]
Use: \( S_n = \frac{n(t_1 + t_n)}{2} \)
Substitute: \( n = 33, t_1 = -2, t_n = 158 \)
\[S_{33} = \frac{33(-2 + 158)}{2}\]
\[S_{33} = 2574\]

9. Use the given data about each arithmetic series to determine the indicated value.

a) \( S_{17} = 106.25 \) and \( t_{17} = 8.25 \); determine \( t_1 \)

Use: \( S_n = \frac{n(t_1 + t_n)}{2} \)
Substitute: \( S_n = 106.25, n = 17, t_n = 8.25 \)
\[106.25 = \frac{17(t_1 + 8.25)}{2}\]
\[212.5 = 17t_1 + 140.25\]
\[17t_1 = 72.25\]
\[t_1 = 4.25\]

b) \( S_{15} = 337.5 \) and \( t_1 = -2 \); determine \( d \)

Use: \( S_n = \frac{n(t_1 + t_n)}{2} \) to determine \( t_{15} \).
Substitute: \( S_n = 337.5, n = 15, t_1 = -2 \)
\[337.5 = \frac{15(-2 + t_{15})}{2}\]
\[675 = -30 + 15t_{15}\]
\[705 = 15t_{15}\]
\[t_{15} = 47\]
Use: \( t_n = t_1 + d(n - 1) \)
Substitute: \( t_n = 47, t_1 = -2, n = 15 \)
\[47 = -2 + d(15 - 1)\]
\[49 = 14d\]
\[d = 3.5\]
1.3 Geometric Sequences

**FOCUS** Solve problems involving geometric sequences.

Get Started

For each sequence below, what are the next 2 terms? What is the rule?

- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
- $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$
- $1, -3, 9, -27, 81, \ldots$

Sample response: For the first sequence, the denominators are the natural numbers. The rule is: add 1 to the denominator; so the next two terms are: $\frac{1}{6}, \frac{1}{7}$

For the second sequence, the denominators are powers of 3. The rule is: multiply the denominator by 3; so the next two terms are: $\frac{1}{243}, \frac{1}{729}$

For the third sequence, the rule is: multiply by $-3$; so the next two terms are: $-243, 729$

Construct Understanding

A French pastry called *mille feuille* or “thousand layers” is made from dough rolled into a square, buttered, and then folded into thirds to make three layers. This process is repeated many times. Each step of folding and rolling is called a turn.

How many turns are required to get more than 1000 layers?

Each turn produces 3 times as many layers, so start with 3 then keep multiplying by 3 until the number of layers is greater than 1000.

<table>
<thead>
<tr>
<th>Number of turns</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of layers</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>243</td>
<td>729</td>
<td>2187</td>
</tr>
</tbody>
</table>

From the table, 7 turns are required to produce more than 1000 layers.
A **geometric sequence** is formed by multiplying each term after the 1st term by a constant, to determine the next term.

For example, $4, 4(3), 4(3)^2, 4(3)^3, \ldots$, is the geometric sequence: $4, 12, 36, 108, \ldots$

The first term, $t_1$, is 4 and the constant is 3.

The constant is the **common ratio**, $r$, of any term after the first, to the preceding term.

The common ratio is any non-zero real number.

To determine the common ratio, divide any term by the preceding term.

For the geometric sequence above:

$$r = \frac{12}{4}, \quad r = \frac{36}{12}, \quad r = \frac{108}{36}$$

$r = 3$  \quad $r = 3$  \quad $r = 3$

The sequence $4, 12, 36, 108, \ldots$, is an **infinite geometric sequence** because it continues forever.

The sequence $4, 12, 36, 108$ is a **finite geometric sequence** because the sequence is limited to a fixed number of terms.

Here are some other examples of geometric sequences.

- This is an **increasing** geometric sequence because the terms are increasing: $2, 10, 50, 250, 1250, \ldots$
  
  The sequence is **divergent** because the terms do not approach a constant value.

- This is a geometric sequence that neither increases, nor decreases because consecutive terms have numerically greater values and different signs: $1, -2, 4, -8, 16, \ldots$
  
  The sequence is **divergent** because the terms do not approach a constant value.

- This is a **decreasing** geometric sequence because the terms are decreasing:
  
  $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$
  
  The sequence is **convergent** because the terms approach a constant value of 0.
Example 1  Determining a Term of a Given Geometric Sequence

a) Determine the 12th term of this geometric sequence: 512, −256, 128, −64, ...

b) Identify the sequence as convergent or divergent.

SOLUTION

a) 512, −256, 128, −64, ...

The common ratio is:

\[ r = \frac{-256}{512} \]

\[ r = -\frac{1}{2} \]

Multiply the first term, 512, by consecutive powers of \(-\frac{1}{2}\):

\[ t_1 = 512 \]

\[ t_2 = 512 \left( -\frac{1}{2} \right) \]

\[ t_3 = 512 \left( -\frac{1}{2} \right)^2 \]

\[ t_4 = 512 \left( -\frac{1}{2} \right)^3 \]

The exponent of each power of the common ratio is 1 less than the term number.

So, the 12th term is:

\[ t_{12} = 512 \left( -\frac{1}{2} \right)^{12-1} \]

\[ t_{12} = 512 \left( -\frac{1}{2} \right)^{11} \]

\[ t_{12} = -\frac{1}{4} \]

The 12th term is \(-\frac{1}{4}\).

b) Since consecutive terms approach a constant value of 0, the sequence is convergent.

THINK FURTHER

In Example 1, how can you predict whether a term is positive or negative?

The sign of the term depends on the sign of the power of \(-\frac{1}{2}\). When this power has an even exponent, the term is positive. When this power has an odd exponent, the term is negative.

Check Your Understanding

1. a) Determine the 10th term of this geometric sequence: 2, −6, 18, −54, ...

b) Identify the sequence as convergent or divergent.

Answers:

1. a) \(-39,366\)

b) divergent
Chapter 1: Sequences and Series

Example 2 Creating a Geometric Sequence

Create a geometric sequence whose 5th term is 48.

SOLUTION

Work backward.

Choose a common ratio that is a factor of 48, such as 2.

Repeatedly divide 48 by 2.

A possible geometric sequence is: 3, 6, 12, 24, 48, ...

THINK FURTHER

In Example 2, why does it make sense to choose a value for \( r \) that is a factor of 48? Could you choose any value for \( r \)?

When \( r \) is a factor of 48, I can use mental math to determine previous terms. No, \( r \) cannot be 0.

A geometric sequence with first term, \( t_1 \), and common ratio, \( r \), can be written as:

\[
\begin{align*}
& t_1, \quad t_1r, \quad t_1r^2, \quad t_1r^3, \quad t_1r^4, \quad \ldots, \quad t_1r^{n-1} \\
& \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
& t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_n
\end{align*}
\]

The exponent of each power of the common ratio is 1 less than the term number.

The General Term of a Geometric Sequence

For a geometric sequence with first term, \( t_1 \), and common ratio, \( r \), the general term, \( t_n \), is:

\[ t_n = t_1r^{n-1} \]

Recall that the product of two negative numbers is positive. So, a square number may be the product of two equal negative numbers or two equal positive numbers. For example, when \( r^2 = 4 \)

then \( r = \pm \sqrt{4} \)

and \( r = 2 \) or \( r = -2 \)

Check Your Understanding

2. Create a geometric sequence whose 6th term is 27.

\( t_6 = 27 \)

Divide by a common ratio that is a factor of 27, such as 3.

\( t_6 = 27; \ t_5 = \frac{27}{3}, \) or 9;

\( t_5 = \frac{9}{3}, \) or 3; \( t_4 = \frac{3}{3}, \) or 1;

\( t_3 = 1; \ t_2 = \frac{1}{3}; \ t_1 = \frac{1}{9} \)

A possible geometric sequence is:

1, 1, 1, 3, 9, 27, ...

2. Create a geometric sequence whose 6th term is 27.

\[ \frac{t_6}{27} = \frac{t_5}{9}; \quad \frac{t_5}{3} \]

or \( t_4 = 3 \); \( t_3 = 1 \); \( t_2 = \frac{1}{3} \); \( t_1 = \frac{1}{9} \)

A possible geometric sequence is:

1, 1, 1, 3, 9, 27, ...

Check Your Understanding

Answer:

2. \( \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, \ldots \)
To determine the common ratio of a geometric sequence, you may need to solve an equation of this form:

\[ r^4 = 81 \]

then \[ r^2 = 9 \]
and \[ r = 3 \text{ or } r = -3 \]

**Example 3**

**Determining Terms and the Number of Terms in a Finite Geometric Sequence**

In a finite geometric sequence, \( t_1 = 5 \) and \( t_5 = 1280 \)

**a)** Determine \( t_3 \) and \( t_6 \).

**b)** The last term of the sequence is 20480. How many terms are in the sequence?

**SOLUTION**

**a)** Determine the common ratio.

Use: \( t_n = t_1 r^{n-1} \)
Substitute: \( n = 5, t_5 = 1280, t_1 = 5 \)

\[ 1280 = 5r^4 \]

Divide each side by 5.

\[ 256 = r^4 \]
Take the fourth root of each side.

\[ r = \pm 4 \] or \( r = 4 \)

There are 2 possible values for \( r \).
When \( r = -4 \), then \( t_3 = 5(-4) = -20 \)
When \( r = 4 \), then \( t_3 = 5(4) = 20 \)

To determine \( t_6 \), use: \( t_n = t_1 r^{n-1} \)
Substitute: \( n = 6, t_1 = 5, r = -4 \)

\[ t_6 = 5(-4)^5 \]

\[ t_6 = -5120 \]

So, \( t_3 = -20 \) or 20, and \( t_6 = -5120 \) or 5120.

**b)** Since the last term is positive, use the positive value of \( r \).

\[ t_n = t_1 r^{n-1} \]
Substitute: \( t_n = 20480, t_1 = 5, r = 4 \)

\[ 20480 = 5R^{n-1} \]
Divide each side by 5.

\[ 4096 = 4^{n-1} \]

Use guess and test to determine which power of 4 is equal to 4096.

Guess: \( 4^4 = 256 \) This is too low.
Guess: \( 4^6 = 4096 \) This is correct.

So, \( 4^n = 4^{n-1} \)
Equate exponents.

\[ 6 = n - 1 \]
\[ n = 7 \]

There are 7 terms in the sequence.

**THINK FURTHER**

Why is \( r^2 \neq -9 \)?

The square of a real number cannot be negative.

**Check Your Understanding**

3. In a finite geometric sequence, \( t_7 = 7 \) and \( t_6 = 567 \)

**a)** Determine \( t_3 \) and \( t_6 \).

**b)** The last term is 45927. How many terms are in the sequence?

**Answers:**

3. **a)** \(-21 \) or 21, \(-1701 \) or 1701
   **b)** 9 terms
Example 4 Using a Geometric Sequence to Solve a Problem

The population of Airdrie, Alberta, experienced an average annual growth rate of about 9% from 2001 to 2006. The population in 2006 was 28 927. Estimate the population in each year to the nearest thousand.

a) 2011

b) 2030, the 125th anniversary of Alberta becoming part of Canada

What assumption did you make? Is this assumption reasonable?

SOLUTION

a) A growth rate of 9% means that each year the population increases by 9%, or 0.09.

The population in 2006 was 28 927.

So, the population in 2007 was:

\[ 28,927 \times 1.09 = 28,927 \times 1.09 \]

Increasing a quantity by 9% is the same as multiplying it by 1.09.

So, to determine a population with a growth rate of 9%, multiply the current population by 1.09.

The annual populations form a geometric sequence with 1st term 28 927 and common ratio 1.09.

The population in 2006 is the 1st term. So, the population in 2011 is the 6th term:

\[ 28,927 \times 1.09^5 \]

The population in 2011 is approximately 45 000.

b) To predict the population in 2030, determine \( n \), the number of years from 2006 to 2030:

\[ n = 2030 - 2006 \]

\[ n = 24 \]

The population in 2030 is:

\[ 28,927 \times 1.09^{24} = 228,843.903 \ldots \]

The population in 2030 will be approximately 229 000.

We assume that the population increase of 9% annually continues. This assumption may be false because the rate of growth may change in future years. This assumption is reasonable for a short time span, but not for a longer time span, such as 100 years.

Check Your Understanding

4. Statistics Canada estimates the population growth of Canadian cities, provinces, and territories. The population of Nunavut is expected to grow annually by 0.8%. In 2009, its population was about 30 000. Estimate the population in each year to the nearest thousand.

a) 2013

b) 2049; Nunavut’s 50th birthday

a) For a growth rate of 0.8%, multiply the current population by 1.008.

The annual populations form a geometric sequence with 1st term 30 000 and common ratio 1.008.

The population in 2009 is the 1st term. So, the population in 2013 is the 5th term:

\[ 30,000 \times 1.008^5 = 30,971.581 \ldots \]

The population in 2013 will be approximately 31 000.

b) Determine \( n \), the number of years from 2009 to 2049:

\[ n = 2049 - 2009 \]

\[ n = 40 \]

The population in 2049 is:

\[ 30,000 \times 1.008^{40} = 41,261.265 \ldots \]

The population in 2049 will be approximately 41 000.

Check Your Understanding Answers:

4. a) approximately 31 000

b) approximately 41 000
Discuss the Ideas

1. How do you determine whether a given sequence is geometric? What assumptions do you make?

After the first term, I divide each term by its preceding term. If these quotients are equal, then the sequence is geometric and the quotient is the common ratio. I assume that the pattern in the terms continues.

2. Which geometric sequences are created when \( r = 1 \)? \( r = -1 \)?

When \( r = 1 \), all terms in the sequence are equal; for example, 3, 3, 3, 3, ...

When \( r = -1 \), the terms have the same numerical value, but alternate in sign; for example, -4, 4, -4, 4, ...

Exercises

A

3. Which sequences could be geometric? If a sequence is geometric, state its common ratio.
      
a) 1, 2, 4, 8, 16, …
      The sequence is geometric. \( r: \frac{2}{1} = 2 \)

b) 4, 9, 16, 25, 36, …
      The sequence is not geometric.

c) -3, 2, 7, 12, 17, …
      The sequence is not geometric.

d) 6, 0.6, 0.06, 0.006, …
      The sequence is geometric. \( r: \frac{0.6}{6} = 0.1 \)

e) 10, 100, 1000, 10 000
      The sequence is geometric. \( r: \frac{100}{10} = 10 \)

f) 2, 4, 6, 8, 10, …
      The sequence is not geometric.
4. State the common ratio, then write the next 3 terms of each geometric sequence.

   a) \(-1, -3, -9, \ldots\)

   \(r = \frac{-3}{-1} = 3\)

   The next 3 terms are: 
   \(-27, -81, -243\)

   b) \(48, 24, 12, \ldots\)

   \(r = \frac{24}{48} = 0.5\)

   The next 3 terms are: 
   \(6, 3, 1.5\)

   c) \(4, -2, 1, \ldots\)

   \(r = \frac{-2}{4} = -0.5\)

   The next 3 terms are: 
   \(-0.5, 0.25, -0.125\)

   d) \(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots\)

   \(r = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}\)

   The next 3 terms are: 
   \(\frac{1}{54}, \frac{1}{162}, \frac{1}{486}\)

5. For each geometric sequence, determine the indicated value.

   a) \(3, 6, 12, \ldots\); determine \(t_7\)

   \(r = \frac{6}{3} = 2\)

   Use: 
   \(t_n = t_1r^{n-1}\)

   Substitute: \(n = 7, t_1 = 3, r = 2\)

   \(t_7 = 3(2)^{7-1} = 192\)

   b) \(18, 9, 4.5, \ldots\); determine \(t_6\)

   \(r = \frac{9}{18} = 0.5\)

   Use: 
   \(t_n = t_1r^{n-1}\)

   Substitute: \(n = 6, t_1 = 18, r = 0.5\)

   \(t_6 = 18(0.5)^{6-1} = 0.5625\)

   c) \(23, -46, 92, \ldots\); determine \(t_{10}\)

   \(r = \frac{-46}{23} = -2\)

   Use: 
   \(t_n = t_1r^{n-1}\)

   Substitute:
   \(n = 10, t_1 = 23, r = -2\)

   \(t_{10} = 23(-2)^{10-1} = -11776\)

   d) \(2, \frac{1}{2}, \frac{1}{4}, \ldots\); determine \(t_5\)

   \(r = \frac{\frac{1}{2}}{2} = \frac{1}{4}\)

   Use: 
   \(t_n = t_1r^{n-1}\)

   Substitute: 
   \(n = 5, t_1 = 2, r = \frac{1}{4}\)

   \(t_5 = 2\left(\frac{1}{4}\right)^{5-1} = \frac{1}{128}\)

6. Write the first 4 terms of each geometric sequence, given the 1st term and the common ratio. Identify the sequence as decreasing, increasing, or neither. Justify your answers.

   a) \(t_1 = -3; r = 4\)

   \(t_1 = -3\)

   \(t_2 = (-3)(4) = -12\)

   \(t_3 = (-12)(4) = -48\)

   \(t_4 = (-48)(4) = -192\)

   The sequence is decreasing because the terms are decreasing.

   b) \(t_1 = 5; r = 2\)

   \(t_1 = 5\)

   \(t_2 = (5)(2) = 10\)

   \(t_3 = (10)(2) = 20\)

   \(t_4 = (20)(2) = 40\)

   The sequence is increasing because the terms are increasing.
7. Write the first 5 terms of each geometric sequence.

a) the 6th term is 64

Sample response: \( t_6 = 64 \)  
Divide by a common ratio that is a factor of 64, such as \(-2\).  
\[
\begin{align*}
t_1 &= 64 \\
t_2 &= \frac{64}{-2} \\
t_3 &= \frac{64}{-2} \\
t_4 &= \frac{64}{-2} \\
t_5 &= 4 \\
t_6 &= 2 \\
\end{align*}
\]
A possible geometric sequence is: 
\(-2, 4, -8, 16, -32, \ldots\)

b) the 1st term is \( \frac{3}{4} \)

Sample response: \( t_1 = \frac{3}{4} \)  
Choose a value for \( r \), such as \( r = 4 \).  
\[
\begin{align*}
t_1 &= \frac{3}{4} \\
t_2 &= \left(\frac{3}{4}\right)4 \\
t_3 &= \left(\frac{3}{4}\right)4 \cdot 4 \\
t_4 &= \left(\frac{3}{4}\right)4 \cdot 4 \cdot 4 \\
t_5 &= \left(\frac{3}{4}\right)4 \cdot 4 \cdot 4 \cdot 4 \\
\end{align*}
\]
A possible geometric sequence is: 
\(\frac{3}{4}, 3, 12, 48, 192, \ldots\)

c) every term is negative

Sample response: Choose a negative 1st term and a positive common ratio, such as \( t_1 = -5 \) and \( r = 2 \).  
\[
\begin{align*}
t_1 &= -5 \\
t_2 &= (-5)(2) \\
t_3 &= (-10)(2) \\
t_4 &= (-20)(2) \\
t_5 &= (-40)(2) \\
t_6 &= (-80) \\
\end{align*}
\]
A possible geometric sequence is: 
\(-5, -10, -20, -40, -80, \ldots\)

d) every term is an even number

Sample response: Choose an even 1st term and an odd or even common ratio, such as \( t_1 = 4 \) and \( r = 3 \).  
\[
\begin{align*}
t_1 &= 4 \\
t_2 &= (4)(3) \\
t_3 &= (12)(3) \\
t_4 &= (36)(3) \\
t_5 &= (108)(3) \\
t_6 &= 324 \\
\end{align*}
\]
A possible geometric sequence is: 
\(4, 12, 36, 108, 324, \ldots\)
8. Use the given data about each finite geometric sequence to determine the indicated values.

a) Given \( t_1 = -1 \) and \( r = -2 \)

i) Determine \( t_9 \).

Use: \( t_n = t_1 r^{n-1} \)

\[ t_9 = (-1)(-2)^8 \]

\[ t_9 = -256 \]

ii) The last term is \(-4096\). How many terms are in the sequence?

Use \( t_n = t_1 r^{n-1} \) to determine \( n \).

Substitute: \( t_n = -4096, t_1 = -1, r = -2 \)

\[ -4096 = (-1)(-2)^{n-1} \] Divide by \(-1\).

\[ 4096 = (-2)^{n-1} \]

\[ (-2)^{13} = (-2)^{n-1} \] Equate exponents.

\[ 13 = n - 1 \]

\[ n = 14 \] There are 14 terms in the sequence.

b) Given \( t_1 = 0.002 \) and \( t_4 = 2 \)

i) Determine \( t_7 \).

Use \( t_n = t_1 r^{n-1} \) to determine the common ratio, \( r \).

Substitute: \( n = 4, t_4 = 2, t_1 = 0.002 \)

\[ 2 = 0.002 r^3 \] Divide each side by 0.002.

\[ 1000 = r^3 \]

\[ r = 10 \]

To determine \( t_7 \), use: \( t_n = t_1 r^{n-1} \)

Substitute: \( n = 7, t_1 = 0.002, r = 10 \)

\[ t_7 = 0.002(10)^6 \]

\[ t_7 = 20000 \]

ii) Determine which term has the value 20 000.

Use \( t_n = t_1 r^{n-1} \) to determine \( n \).

Substitute: \( t_n = 20000, t_1 = 0.002, r = 10 \)

\[ 20000 = 0.002(10)^{n-1} \]

\[ 1000000 = 10^n \]

\[ 10^3 = 10^n \]

\[ 3 = n - 1 \]

\[ n = 4 \]

20 000 is the 8th term.
9. a) Insert three numbers between 8 and 128, so the five numbers form an arithmetic sequence. Explain what you did.

The sequence has the form: 8, 8 + d, 8 + 2d, 8 + 3d, 128
Write \(128 = 8 + 4d\), then solve for \(d\) to get \(d = 30\).
The arithmetic sequence is: 8, 38, 68, 98, 128

b) Insert three numbers between 8 and 128, so the five numbers form a geometric sequence. Explain what you did.

The sequence has the form: 8, 8r, 8r^2, 8r^3, 128
Write \(128 = 8r^4\), then solve for \(r\) to get \(r^4 = 16\), so \(r = 2\) or \(-2\).
The geometric sequence is: 8, 16, 32, 64, 128;
or \(8, -16, 32, -64, 128\)

c) What was similar about your strategies in parts a and b? What was different?

For each sequence, I wrote an equation for the 5th term, then solved the equation to determine the common difference and common ratio. For the arithmetic sequence, I added the common difference to get the next terms. For the geometric sequences, there were two possible common ratios, and I multiplied by each common ratio to get the next terms.

10. Suppose a person is given 1¢ on the first day of April; 3¢ on the second day; 9¢ on the third day, and so on. This pattern continues throughout April.

a) About how much money will the person be given on the last day of April?

There are 30 days in April.
The daily amounts, in cents, form this geometric sequence:
1, 1(3), 1(3)^2, \ldots, 1(3)^{29}
The amount on the last day, in cents, is \(1(3)^{29} = 6.863 \times 10^{13}\)
Divide by 100 to convert the amount to dollars:
approximately \(6.863 \times 10^{11}\)

b) Why might it be difficult to determine the exact amount using a calculator?

A calculator screen shows only 10 digits, and the number of digits in the amount of money in dollars is greater than 10.
11. In a geometric sequence, \( t_3 = 9 \) and \( t_6 = 1.125 \); determine \( t_7 \) and \( t_9 \).

Use \( t_n = t_1 r^{n-1} \) twice to get two equations.

For \( t_3 \), substitute: \( n = 3, t_3 = 9 \)

\[
9 = t_1 r^{3-1} \\
9 = t_1 r^2 \quad \Box
\]

For \( t_6 \), substitute: \( n = 6, t_6 = 1.125 \)

\[
1.125 = t_1 r^{6-1} \\
1.125 = t_1 r^5 \quad \Box
\]

Write equation \( \Box \) as: \( 1.125 = t_1 r^2(r^3) \)

From equation \( \Box \), substitute \( t_1 r^2 = 9 \)

\[
1.125 = 9r^3 \\
\text{Divide by 9.}
\]

\[
0.125 = r^3 \\
r = \sqrt[3]{0.125} \\
r = 0.5
\]

So, \( t_7 = t_1 r^6 \) and \( t_9 = t_1 r^8 \)

\[
= 1.125 (0.5) = 0.5625 (0.5)^2 \\
= 0.5625 \times 0.140625
\]

12. An arithmetic sequence and a geometric sequence have the same first term. The common difference and common ratio are equal and greater than 1. Which sequence increases more rapidly as more terms are included? Use examples to explain.

This arithmetic sequence has \( t_1 = 3 \) and \( d = 4 \):

\[3, 7, 11, 15, 19, 23, \ldots\]

This geometric sequence has \( t_1 = 3 \) and \( r = 4 \):

\[3, 12, 48, 192, 768, 3072, \ldots\]

The geometric sequence increases more rapidly because we are multiplying instead of adding to get the next term.

13. A ream of paper is about 2 in. thick. Imagine a ream of paper that is continually cut in half and the two halves stacked one on top of the other. How many cuts have to be made before the stack of paper is taller than 318 ft., the height of Le Chateau York in Winnipeg, Manitoba?

Let the number of cuts be \( n \).

The heights of the stacks of paper, in inches, are a geometric sequence with 1st term 2 and common ratio 2:

\[2, 2(2), 2(2)^2, 2(2)^3, \ldots, 2(2)^n\]

Write 318 ft. in inches: 318(12 in.) = 3816 in.

Write an equation:

\[2(2)^n = 3816 \quad \text{Solve for } n.\]

\[2^n = 1908\]

Use guess and test: \( 2^{10} = 1024; 2^{11} = 2048 \)

10 cuts will not be enough.

11 cuts will produce a stack that is: \( 2(2)^{11} \text{ in.} = 4096 \text{ in. high} \)

11 cuts have to be made.
14. Between the Canadian censuses in 2001 and 2006, the number of people who could converse in Cree had increased by 7%. In 2006, 87,285 people could converse in Cree. Assume the 5-year increase continues to be 7%. Estimate to the nearest hundred how many people will be able to converse in Cree in 2031.

To model a growth rate of 7%, multiply by 1.07.
The number of people every 5 years form a geometric sequence with first term 87,285 and common ratio 1.07.
Every 5 years is: 2006, 2011, 2016, 2021, 2026, 2031, ...
So, the number of people in 2031 is:
87,285(1.07)^5 = 122,421.7278...
The number of people who will be able to converse in Cree in 2031 will be approximately 122,400.

15. A farmer in Saskatchewan wants to estimate the value of a new combine after several years of use. A new combine worth $370,000 depreciates in value by about 10% each year.

a) Estimate the value of the combine at the end of each of the first 5 years. Write the values as a sequence.

When the value decreases by 10%, the new value is 90% of the original value.
To determine a depreciation value of 10%, multiply by 0.9.
The values, in dollars, at the end of each of the first 5 years are:
370,000(0.9), 370,000(0.9)^2, 370,000(0.9)^3, 370,000(0.9)^4, 370,000(0.9)^5
The values, to the nearest dollar, are: $333,000, $299,700, $269,730, $242,757, $218,481

b) What type of sequence did you write in part a? Explain your reasoning.

The sequence is geometric because I multiplied by a constant to get each value from the preceding value.

c) Predict the value of the combine at the end of 10 years.

At the end of 10 years, to the nearest dollar, the value is:
$370,000(0.9)^{10} = $129,011

16. a) Show that squaring each term in a geometric sequence produces the same type of sequence. What is the common ratio?

Consider the sequence: t_1, t_2, t_3, t_4, t_5, ..., t_n^2
Square each term. The new sequence is:
t_1^2, t_2^2, t_3^2, t_4^2, t_5^2, ..., t_n^{2n-2}
This is a geometric sequence with 1st term t_1^2 and common ratio r^2.
b) Show that raising each term in a geometric sequence to the $m$th power of each term produces the same type of sequence. What is the common ratio?

Consider the sequence: $t_n, t_n, t_n, t_n, \ldots, t_n, t_n$
Raise each term to the $m$th power.
The new sequence is: $t_n^m, t_n^m, t_n^m, t_n^m, \ldots, t_n^m, t_n^m$
This is a geometric sequence with 1st term $t_1^m$ and common ratio $r^m$.

Multiple-Choice Questions

1. Which expression below represents the $n$th term of this geometric sequence?
   \[ 9, -6, 4, -\frac{8}{3}, \ldots \]
   A. $9 \left( \frac{2}{3} \right)^{n-1}$  B. $9 \left( -\frac{2}{3} \right)^{n-1}$  C. $\frac{2}{3} (9^{n-1})$  D. $-\frac{2}{3} (9^{n-1})$

2. Which geometric sequence does not have a common ratio of $-0.5$?
   A. $-5, 2.5, -1.25, 0.625, \ldots$  B. $6, -3, 1.5, -0.75, \ldots$
   C. $\frac{1}{200}, -\frac{1}{100}, \frac{1}{50}, -\frac{1}{25}, \ldots$  D. $-\frac{1}{3}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{24}, \ldots$

3. The value of a car in each of its first 3 years is: $24\,000, 20\,400, 17\,340$
   These amounts form a sequence.
   Which statement describes this sequence?
   A. arithmetic with common difference $3600$
   B. geometric with common ratio approximately 1.18
   C. geometric with common ratio 0.85
   D. neither arithmetic nor geometric

Study Note

Can a sequence be both arithmetic and geometric? Explain.

A sequence can be both arithmetic and geometric when all the terms are equal. For example: $3, 3, 3, 3, \ldots$ is an arithmetic sequence with 1st term 3 and common difference 0; and $3, 3, 3, 3, \ldots$ is a geometric sequence with 1st term 3 and common ratio 1.

ANSWERS
3. a) 2  d) 0.1  e) 10  4. a) 3; -27, -81, -243  b) 0.5; 6, 3, 1.5  c) -0.5; -0.5, 0.25, -0.125  d) $\frac{1}{3}, \frac{1}{3}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$
   5. a) 192  b) 0.5625  c) -11 776
   d) $\frac{1}{128}$  6. a) -3, -12, -48, -192; decreasing  b) 5, 10, 20, 40; increasing
   c) -0.5, 1.5, -4.5, 13.5; neither  d) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$; decreasing
   8. a) i) -256  ii) 13  b) i) 2000  ii) $t_4$  9. a) 8, 38, 68, 98, 128, \ldots
   b) 8, -16, 32, -64, 128, \ldots or 8, 16, 32, 64, 128, \ldots  10. a) approximately $6.863 \times 10^{11}$
   11. 0.5625; 0.140 625  12. geometric  13. 11 cuts  14. approximately 122 400 people
   15. a) $333\,000, 299\,700, 269\,730, 242\,757, 218\,481$  b) geometric  c) $129\,011$

Multiple Choice
1.4 Geometric Series

**FOCUS** Derive a rule to determine the sum of \( n \) terms of a geometric series, then solve related problems.

**Get Started**

Two geometric sequences have the same first term but the common ratios are opposite integers. Which corresponding terms are equal? Which corresponding terms are different? Use an example to explain.

Suppose the 1st term is 3 and the common ratios are 2 and \(-2\).

One sequence is: 3, 3(2), 3(2)^2, 3(2)^3, \ldots; or 3, 6, 12, 24, \ldots

The other sequence is: 3, 3(-2), 3(-2)^2, 3(-2)^3, \ldots; or 3, -6, 12, -24, 48, \ldots

The odd numbered terms are equal and the even numbered terms are different.

**Construct Understanding**

Caitlan traced her direct ancestors, beginning with her 2 parents, 4 grandparents, 8 great-grandparents, and so on.

```
Caitlan Wen Shaan
  ♀ Cheung

Timothy Yalmann
  ♂ Cheung

Soo-Ann Yong Cheung
  ♀ 1st generation

Kan Shu
  ♂ Cheung

Mei Lin
  ♀ Cheung

Yeu Jian
  ♂ Yong

Mi Lan
  ♀ Yong 2nd generation
```

Determine the total number of Caitlan’s direct ancestors in 20 generations.

The numbers in each generation form a geometric sequence with common ratio 2: 2, 4, 8, 16, \ldots; or 2^1, 2^2, 2^3, 2^4, \ldots

Make a table for the total number of direct ancestors and look for patterns.

<table>
<thead>
<tr>
<th>Generation</th>
<th>Number of Ancestors</th>
<th>Total Ancestors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>
The Sum of $n$ Terms of a Geometric Series

For the geometric series $t_1 + t_1r + t_1r^2 + \ldots + t_1r^{n-1}$, the sum of $n$ terms, $S_n$, is:

$$S_n = \frac{t_1(1 - r^n)}{1 - r}, \quad r \neq 1$$
Example 1  Determining the Sum of Given Terms of a Geometric Series

Determine the sum of the first 15 terms of this geometric series:

\[ 40 - 20 + 10 - 5 + 2.5 - \ldots \]

Write the sum to the nearest hundredth.

**SOLUTION**

\[ t_1 = 40 \text{ and } r = \frac{-20}{40} = \frac{-1}{2} \]

Use: \[ S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \]

Substitute: \[ n = 15, \ t_1 = 40, \ r = \frac{-1}{2} \]

\[ S_{15} = \frac{40\left(1 - \left(\frac{-1}{2}\right)^{15}\right)}{1 - \left(\frac{-1}{2}\right)} \]

Use a calculator.

\[ S_{15} = 26.6674 \ldots \]

The sum of the first 15 terms is approximately 26.67.

Example 2  Determining Terms of a Geometric Series

The sum of the first 10 terms of a geometric series is \(-29\,524\).
The common ratio is \(-3\). Determine the 1st term.

**SOLUTION**

Suppose the geometric series has 1st term, \(t_1\), and common ratio, \(r\).

Use: \[ S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \]

Substitute: \[ n = 10, \ S_n = -29\,524, \ r = -3 \]

\[ -29\,524 = \frac{t_1(1 - (-3)^{10})}{1 - (-3)} \]

**Check Your Understanding**

1. Determine the sum of the first 12 terms of this geometric series:

   \[ 3 + 12 + 48 + 192 + \ldots \]

   \[ t_1 = 3 \text{ and } r = \frac{12}{3} = 4 \]

   Use: \[ S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \]

   Substitute: \(n = 12, \ t_1 = 3, \ r = 4\)

   \[ S_{12} = \frac{3(1 - 4^{12})}{1 - 4} \]

   \[ S_{12} = 16\,777\,215 \]

   The sum of the first 12 terms is 16\,777\,215.

2. The sum of the first 14 terms of a geometric series is 16\,383.
The common ratio is \(-2\). Determine the 1st term.

   Use: \[ S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \]

   Substitute: \(n = 14, \ S_n = 16\,383, \ r = -2\)

   \[ \checkmark \]

**Answers:**

1. \(16\,777\,215\)  2. \(-3\)
Check Your Understanding

3. Calculate the sum of this geometric series:
   
   \(-3 - 15 - 75 - \ldots - 46875\)

   \(t_1 = -3\) and \(r = \frac{-15}{-3} = 5\)
   
   To determine \(n\), use: \(t_n = t_1r^{n-1}\)
   
   Substitute: \(t_n = -46875\),
   
   \(t_1 = -3, r = 5\)
   
   \(-46875 = (-3)(5)^{n-1}\)
   
   \(15625 = 5^{n-1}\)
   
   \(5 = 5^{n-1}\)
   
   \(6 = n - 1\)
   
   \(n = 7\)
   
   There are 7 terms in the series.

   Use: \(S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1\)

   Substitute: \(n = 7, t_1 = -3, r = 5\)

   \(S_7 = \frac{(-3)(1 - 5^7)}{1 - 5}\)

   \(S_7 = -58\,593\)

   The sum is \(-58\,593\).

---

Example 3

**Determining the Sum of a Geometric Series**

Calculate the sum of this geometric series:

\[6 + 12 + 24 + 48 + \ldots + 12\,288\]

**SOLUTION**

\[6 + 12 + 24 + 48 + \ldots + 12\,288\]

\(t_1 = 6\) and \(r = \frac{12}{6} = 2\)

Determine \(n\), the number of terms in the series.

Use: \(t_n = t_1r^{n-1}\)  
Substitute: \(t_n = 12\,288, t_1 = 6, r = 2\)

\(2048 = 2^{n-1}\)  
Simplify. Divide each side by 6.

\(2^{11} = 2^{n-1}\)  
Use guess and test to write 2048 as a power of 2.

\(11 = n - 1\)  
Equate exponents.

\(n = 12\)

There are 12 terms in the series.

Use: \(S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1\)  
Substitute: \(n = 12, t_1 = 6, r = 2\)

\(S_{12} = \frac{6(1 - 2^{12})}{1 - 2}\)

\(S_{12} = 24\,570\)

The sum is 24 570.
Example 4  Using a Geometric Series to Model and Solve a Problem

A person takes tablets to cure an ear infection. Each tablet contains 200 mg of an antibiotic. About 12% of the mass of the antibiotic remains in the body when the next tablet is taken. Determine the mass of antibiotic in the body after each number of tablets has been taken.

a) 3 tablets  

b) 12 tablets

SOLUTION

a) Determine the mass of the antibiotic in the body for 1 to 3 tablets.

<table>
<thead>
<tr>
<th>Number of tablets</th>
<th>Mass of antibiotic (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>200 + 200(0.12) = 224</td>
</tr>
<tr>
<td>3</td>
<td>200 + 200(0.12) + 200(0.12)^2 = 226.88</td>
</tr>
</tbody>
</table>

The problem can be modelled by a geometric series, which has 3 terms because 3 tablets were taken:

200 + 200(0.12) + 200(0.12)^2

The sum is 226.88. So, after taking the 3rd tablet, the total mass of antibiotic in the person’s body is 226.88 mg or just under 227 mg.

b) Determine the sum of a geometric series whose terms are the masses of the antibiotic in the body after 12 tablets.

The series is:

200 + 200(0.12) + 200(0.12)^2 + 200(0.12)^3 + \ldots + 200(0.12)^{11}

Use: 

\[ S_n = \frac{t_1(1 - r^n)}{1 - r}, \quad r \neq 1 \]

Substitute: \( n = 12, \quad t_1 = 200, \quad r = 0.12 \)

\[ S_{12} = \frac{200(1 - 0.12^{12})}{1 - 0.12} \]

\[ S_{12} = 227.2727\ldots \]

The mass of antibiotic in the body after 12 tablets is approximately 227.27 mg, or just over 227 mg.

THINK FURTHER

In Example 4, compare the masses of antibiotic remaining in the body for parts a and b. What do you notice about the masses?

For each extra tablet, the increase in mass is less than the preceding increase. The masses seem to be approaching a constant value that is slightly greater than 227 mg.

Check Your Understanding

4. A person takes tablets to cure a chest infection. Each tablet contains 500 mg of an antibiotic. About 15% of the mass of the antibiotic remains in the body when the next tablet is taken. Determine the mass of antibiotic in the body after each number of tablets:

a) 3 tablets  

b) 10 tablets

Answers:

a) 

<table>
<thead>
<tr>
<th>Number of tablets</th>
<th>Mass of antibiotic (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>500</td>
</tr>
<tr>
<td>2</td>
<td>500 + 500(0.15) = 575</td>
</tr>
<tr>
<td>3</td>
<td>500 + 500(0.15) + 500(0.15)^2 = 586.25</td>
</tr>
</tbody>
</table>

The sum is 586.25. So, after taking the 3rd tablet, the total mass is 586.25 mg, or about 586 mg.

b) Determine the sum of this geometric series:

500 + 500(0.15) + 500(0.15)^2 + 500(0.15)^3 + \ldots + 500(0.15)^{10}

Use: 

\[ S_n = \frac{t_1(1 - r^n)}{1 - r}, \quad r \neq 1 \]

Substitute: \( n = 10, \quad t_1 = 500, \quad r = 0.15 \)

\[ S_{10} = \frac{500(1 - 0.15^{10})}{1 - 0.15} \]

\[ S_{10} = 588.2352\ldots \]

The mass of antibiotic is about 588 mg.

Check Your Understanding

Answers:

4. a) 586.25 mg, or just over 586 mg

b) approximately 588.24 mg, or just over 588 mg
Discuss the Ideas

1. Why do the terms in some geometric series alternate between positive and negative numbers, but the terms in an arithmetic series never alternate?

   In a geometric series, when \( r \) is negative, the terms alternate in sign because powers of \( r \) with an even exponent are positive, and powers of \( r \) with an odd exponent are negative. In an arithmetic series, when \( d \) is negative, the terms decrease; when \( d \) is positive, the terms increase; and when \( d \) is 0 the terms are constant. So, the terms never alternate in sign.

2. How can you identify when a problem may be modelled by an arithmetic series or modelled by a geometric series?

   If a problem involves a number and a constant that is repeatedly added or subtracted, the problem may be modelled by an arithmetic series. If the problem involves a number and a constant that is repeatedly multiplied or divided, then the problem may be modelled by a geometric series.

Exercises

A

3. Write a geometric series for each geometric sequence.

   a) 1, 4, 16, 64, 256, \ldots
   \[ 1 + 4 + 16 + 64 + 256 + \ldots \]

   b) 20, -10, 5, -2.5, 1.25, \ldots
   \[ 20 - 10 + 5 - 2.5 + 1.25 - \ldots \]

4. Which series appear to be geometric? If the series could be geometric, determine \( S_5 \).

   a) \[ 2 + 4 + 8 + 16 + 32 + \ldots \]
   The series could be geometric. \[ S_5 : 2 + 4 + 8 + 16 + 32 = 62 \]

   b) \[ 2 - 4 + 8 - 16 + 32 - \ldots \]
   The series could be geometric. \[ S_5 : 2 - 4 + 8 - 16 + 32 = 22 \]

   c) \[ 1 + 4 + 9 + 16 + 25 + \ldots \]
   The series is not geometric.

   d) \[ -3 + 9 - 27 + 81 - 243 + \ldots \]
   The series could be geometric. \[ S_5 : -3 + 9 - 27 + 81 - 243 = -183 \]
5. Use the given data about each geometric series to determine the indicated value. Give the answers to 3 decimal places where necessary.
   
   a) \( t_1 = 1, \ r = 0.3 \); determine \( S_n \)
   
   Use: \( S_n = \frac{t_1(1-r^n)}{1-r}, \ r \neq 1 \)
   
   Substitute:
   
   \( n = 8, \ t_1 = 1, \ r = 0.3 \)
   
   \( S_8 = \frac{1(1-0.3^8)}{1-0.3} \)
   
   \( S_8 = 1.428 \)
   
   b) \( t_1 = \frac{3}{4}, \ r = \frac{1}{2} \); determine \( S_n \)
   
   Use: \( S_n = \frac{t_1(1-r^n)}{1-r}, \ r \neq 1 \)
   
   Substitute:
   
   \( n = 6, \ t_1 = \frac{3}{4}, \ r = \frac{1}{2} \)
   
   \( S_6 = \frac{\frac{3}{4}(1-(\frac{1}{2})^6)}{1-\frac{1}{2}} \)
   
   \( S_6 = 45 \div 32 \) or approximately 1.406

6. Determine \( S_n \) for each geometric series.
   
   a) \( 2 + 10 + 50 + \ldots \)
   
   \( t_1 = 2 \) and \( r = \frac{10}{2} = 5 \)
   
   Use: \( S_n = \frac{t_1(1-r^n)}{1-r}, \ r \neq 1 \)
   
   Substitute:
   
   \( n = 6, \ t_1 = 2, \ r = 5 \)
   
   \( S_6 = \frac{2(1-5^6)}{1-5} \)
   
   \( S_6 = 7812 \)
   
   b) \( 80 - 40 + 20 - \ldots \)
   
   \( t_1 = 80 \) and \( r = -\frac{40}{80} = -0.5 \)
   
   Use: \( S_n = \frac{t_1(1-r^n)}{1-r}, \ r \neq 1 \)
   
   Substitute:
   
   \( n = 6, \ t_1 = 80, \ r = -0.5 \)
   
   \( S_6 = \frac{80(1-(-0.5)^6)}{1-(-0.5)} \)
   
   \( S_6 = 52.5 \)

7. Determine \( S_{10} \) for each geometric series. Give the answers to 3 decimal places.
   
   a) \( 0.1 + 0.01 + 0.001 + 0.0001 + \ldots \)
   
   \( t_1 = 0.1 \) and \( r = \frac{0.01}{0.1} = 0.1 \)
   
   Use: \( S_n = \frac{t_1(1-r^n)}{1-r}, \ r \neq 1 \)
   
   Substitute:
   
   \( n = 10, \ t_1 = 0.1, \ r = 0.1 \)
   
   \( S_{10} = \frac{0.1(1-0.1^{10})}{1-0.1} \)
   
   \( S_{10} = 0.111 \)
   
   b) \( 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots \)
   
   \( t_1 = 1 \) and \( r = -\frac{1}{3} \)
   
   Use: \( S_n = \frac{t_1(1-r^n)}{1-r}, \ r \neq 1 \)
   
   Substitute:
   
   \( n = 10, \ t_1 = 1, \ r = -\frac{1}{3} \)
   
   \( S_{10} = \frac{1(1-(-\frac{1}{3})^{10})}{1-(-\frac{1}{3})} \)
   
   \( S_{10} = 0.750 \)

8. a) Explain why this series appears to be geometric:

\[ 1 + 5 + 25 + 125 + \ldots \]

After the 1st term, each term is 5 times as great as the preceding term.

**TEACHER NOTE**

**Achievement Indicator**

Questions 5, 6, and 7 address AI 10.6: Determine \( t_n, r, n, \) or \( S_n \) in a problem that involves a geometric series.

**Achievement Indicator**

Question 8 addresses AI 10.1: Identify assumptions made when identifying a geometric series.
b) What information do you need to be certain that this is a geometric series?
   I need to know that the series has a common ratio of 5.

c) What assumptions do you make when you identify or extend a geometric series?
   I assume that the ratio of consecutive terms is the common ratio.

9. For each geometric series, determine how many terms it has then calculate its sum.

   a) \(1 - 2 + 4 - 8 + \ldots - 512\)
   \(t_1 = 1\) and \(r = \frac{-2}{1} = -2\)
   To determine \(n\), use: \(t_n = t_1 r^{n-1}\)
   Substitute: \(t_n = -512, \quad t_1 = 1, \quad r = -2\)
   \[-512 = 1(-2)^{n-1}\]
   \((-2)^9 = (-2)^{n-1}\)
   \[9 = n - 1\]
   \[n = 10\]
   There are 10 terms.
   To determine the sum, use: \(S_n = \frac{t_1(1 - r^n)}{1 - r}, \quad r \neq 1\)
   Substitute: \(n = 10, \quad t_1 = 1, \quad r = -2\)
   \[S_{10} = \frac{1((1 - (-2)^{10}))}{1 - (-2)}\]
   \[S_{10} = -341\]
   The sum is \(-341\).

   b) \(-6561 + 2187 - 729 + 243 - \ldots - 1\)
   \(t_1 = -6561\) and \(r = \frac{2187}{-6561} = -\frac{1}{3}\)
   To determine \(n\), use: \(t_n = t_1 r^{n-1}\)
   Substitute: \(t_n = -1, \quad t_1 = -6561, \quad r = -\frac{1}{3}\)
   \[-1 = -6561\left(-\frac{1}{3}\right)^{n-1}\]
   \[\frac{1}{6561} = \left(-\frac{1}{3}\right)^{n-1}\]
   \[\left(-\frac{1}{3}\right)^8 = \left(-\frac{1}{3}\right)^{n-1}\]
   \[8 = n - 1\]
   \[n = 9\]
   There are 9 terms.
   Use: \(S_n = \frac{t_1(1 - r^n)}{1 - r}, \quad r \neq 1\)
   Substitute: \(n = 9, \quad t_1 = -6561, \quad r = -\frac{1}{3}\)
   \[S_9 = \frac{-6561(1 - (-\frac{1}{3})^9)}{1 - (-\frac{1}{3})}\]
   \[S_9 = -4921\]
   The sum is \(-4921\).
10. Identify the terms in each partial sum of a geometric series.
   \[a) \ S_5 = 62, \ r = 2 \]
   \[b) \ S_n = 1111.1111; \ r = 0.1 \]

   To determine \( t_n \), use the formula:
   \[ S_n = t_1 \left( \frac{1 - r^n}{1 - r} \right), \ r \neq 1 \]

   Substitute: \( n = 5, \ S_n = 62, \ r = 2 \)
   \[ 62 = t_1 \left( \frac{1 - 2^5}{1 - 2} \right) \]
   \[ 62 = 31t_1 \]
   \[ t_1 = 2 \]
   So, \( t_2 = 4 \); \( t_3 = 8 \);
   \( t_4 = 16 \); \( t_5 = 32 \)

   To determine \( t_n \), use the formula:
   \[ S_n = t_1 \left( \frac{1 - r^n}{1 - r} \right), \ r \neq 1 \]

   Substitute: \( n = 8, \ S_n = 1111.1111, \ r = 0.1 \)
   \[ 1111.1111 = t_1 \left( \frac{1 - 0.1^8}{1 - 0.1} \right) \]
   \[ 1111.1111 = 1111.1111 \]
   \[ t_1 = 1000 \]
   So, \( t_2 = 1000(0.1) = 100 \);
   \( t_3 = 100(0.1) = 10 \);
   \( t_4 = 10(0.1) = 1 \);
   \( t_5 = 1(0.1) = 0.1 \);
   \( t_6 = (0.1)(0.1) = 0.01 \);
   \( t_7 = 0.01(0.1) = 0.001 \);
   \( t_8 = 0.001(0.1) = 0.0001 \)

11. On Monday, Ian had 3 friends visit his personal profile on a social networking website. On Tuesday, each of these 3 friends had 3 different friends visit Ian’s profile. On Wednesday, each of the 9 friends on Tuesday had 3 different friends visit Ian’s profile.

   a) Write the total number of friends who visited Ian’s profile as a geometric series. What is the first term? What is the common ratio?

      The 1st term is 3, the number of friends on Monday.
      The common ratio is 3.
      So, the geometric series is: \[ 3 + 9 + 27 \]

   b) Suppose this pattern continued for 1 week. What is the total number of people who visited Ian’s profile? How do you know your answer is correct?

      The geometric series continues and has 7 terms;
      one for each day of the week.
      The series is: \[ 3 + 9 + 27 + 81 + 243 + 729 + 2187 \]

      Use: \[ S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \]
      Substitute: \( n = 7, \ t_1 = 3, \ r = 3 \)
      \[ S_7 = \frac{3(1 - 3^n)}{1 - 3} \]
      \[ S_7 = 3279 \]
      I checked my answer by using a calculator to add the seven terms.
12. Each stroke of a vacuum pump extracts 5% of the air in a 50-m$^3$ tank. How much air is removed after 50 strokes? Give the answer to the nearest tenth of a cubic metre.

<table>
<thead>
<tr>
<th>Number of strokes</th>
<th>Volume removed</th>
<th>Volume remaining</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50(0.05) = 2.5</td>
<td>50(0.95) = 47.5</td>
</tr>
<tr>
<td>2</td>
<td>47.5(0.05) = 2.375, or 2.5(0.95)</td>
<td>47.5(0.95) = 45.125</td>
</tr>
<tr>
<td>3</td>
<td>45.125(0.05) = 2.256 25, or 2.5(0.95)$^2$</td>
<td>45.125(0.95) = 42.625</td>
</tr>
</tbody>
</table>

The volumes removed form a geometric series with 1st term 2.5 and common ratio 0.95.

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 50, t_1 = 2.5, r = 0.95$

$S_{50} = \frac{2.5(1 - 0.95^{50})}{1 - 0.95}$

$S_{50} = 46.1527...$

After 50 strokes, about 46.2 m$^3$ of air is removed.

13. The sum of the first 10 terms of a geometric series is 1705. The common ratio is $-2$. Determine $S_{11}$. Explain your reasoning.

To determine $t_n$, use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $S_n = 1705, n = 10, r = -2$

$1705 = \frac{t_1(1 - (-2)^{10})}{1 - (-2)}$

$1705 = -341t_1$

$t_1 = -5$

Then, $S_{11} = S_{10} + t_{11}$

$S_{11} = 1705 + (-5)(-2)^{10}$

$S_{11} = -3415$

14. Binary notation is used to represent numbers on a computer. For example, the number 1111 in base two represents $1(2)^3 + 1(2)^2 + 1(2)^1 + 1$, or 15 in base ten.

a) Why is the sum above an example of a geometric series?

Each term is one-half of the preceding term.
b) Which number in base ten is represented by 111 111 111 111 111 111 111 in base two? Explain your reasoning.

There are twenty 1s digits in the number, so it can be written as the geometric series:

\[ 1(2)^{19} + 1(2)^{18} + \ldots + 1(2)^1 + 1 \]

This series has 20 terms, with 1st term \(2^{19}\) and common ratio 0.5.

Use: \( S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1 \)

Substitute: \( n = 20, t_1 = 2^{19}, r = 0.5 \)

\[ S_{20} = \frac{2^{19}(1 - 0.5^{20})}{1 - 0.5} \]
\[ S_{10} = 1 \ 048 \ 575 \]

The number is 1 048 575.

15. Show how you can use geometric series to determine this sum:

\[ 1 + 2 + 3 + 4 + 8 + 9 + 16 + 27 + 32 + 64 + 81 + 128 + 243 + 256 + 512 \]

This sum comprises two geometric series:

\[ 1 + 3 + 9 + 27 + 81 + 243 \]
\[ 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 \]

For the first series

Use: \( S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1 \)

Substitute: \( n = 6, t_1 = 1, r = 3 \)

\[ S_6 = \frac{1(1 - 3^6)}{1 - 3} \]
\[ S_6 = 364 \]

For the second series

Use: \( S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1 \)

Substitute: \( n = 9, t_1 = 2, r = 2 \)

\[ S_9 = \frac{2(1 - 2^9)}{1 - 2} \]
\[ S_9 = 1022 \]

The sum is: 364 + 1022 = 1386

16. Determine the common ratio of a geometric series that has these partial sums: \( S_1 = -\frac{49}{8}, S_2 = -\frac{105}{16}, S_3 = -\frac{217}{32} \)

\[ S_n = S_{n-1} + t_n \]

Substitute for \( S_n \) and \( S_{n-1} \).

\[ -\frac{105}{16} = -\frac{49}{8} + t_4 \]
\[ t_4 = -\frac{7}{16} \]

\[ S_n = S_{n-1} + t_n \]

Substitute for \( S_n \) and \( S_{n-1} \).

\[ -\frac{217}{32} = -\frac{105}{16} + t_5 \]
\[ t_5 = -\frac{7}{32} \]
\[ t_5 = t_4(r) \]
\[ -\frac{7}{32} = -\frac{7}{16}(r) \]
\[ r = \frac{1}{2} \]

The common ratio is \( \frac{1}{2} \).
Multiple-Choice Questions

1. For which geometric series is \(-1023\) the sum to 10 terms?
   A. \(1 - 2 + 4 - 8 + \cdots\)
   B. \(1 + 2 + 4 + 8 + \cdots\)
   C. \(-1 + 2 - 4 + 8 - \cdots\)
   D. \(-1 - 2 - 4 - 8 - \cdots\)

2. The sum of the first \(n\) terms of a geometric series is: \(S_n = 4^n - 1\)
   For this series:
   I. The common ratio is 4.
   II. The first 3 terms are 3, 12, and 48.
   III. \(S_{2n} = 2^{4n} - 1\)
   (A) Statements I, II, and III are correct.
   (B) Statements I and II are correct.
   (C) Statements II and III are correct.
   (D) Statements I and III are correct.

Study Note

The rule for the sum of the first \(n\) terms of a geometric series has the restriction \(r \neq 1\). Identify the geometric series with first term \(a\) and \(r = 1\), then determine the sum of \(n\) terms.

\[\text{When the first term is } a \text{ and } r = 1, \text{ the series is } a + a + a + a + \cdots\]
\[\text{The sum of } n \text{ terms is } an.\]

ANSWERS

3. a) \(1 + 4 + 16 + 64 + 256 + \cdots\)  b) \(20 + 10 + 5 - 2.5 + 1.25 - \cdots\)
4. a) 62  b) 22  c) \(-183\)  d) \(-1.428\)
5. a) approximately 1.428  b) approximately 1.406
6. a) 7812  b) 52.5  c) approximately 0.111  d) approximately 0.750
9. a) 10 terms; \(-341\)  b) 9 terms; \(-492\)
   b) 1000, 100, 10, 1, 0.1, 0.01, 0.001, 0.0001
11. a) \(3 + 9 + 27; 3\)  b) 3279 people
12. 46.2 \(m^3\)  13. \(-3415\)  14. b) 1048575  15. 1386  16. \(\frac{1}{2}\)

Multiple Choice
1. D  2. A
CHECKPOINT 2

Self-Assess

<table>
<thead>
<tr>
<th>Can you . . .</th>
<th>To check, try question . . .</th>
<th>For review, see . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>write a geometric sequence and explain how you know it is geometric?</td>
<td></td>
<td>Page 32 in Lesson 1.3 (Example 2)</td>
</tr>
<tr>
<td>identify the assumptions you make when you identify a geometric sequence or series?</td>
<td></td>
<td>Page 35 in Lesson 1.3 (Discuss the Ideas)</td>
</tr>
<tr>
<td>use a rule to determine the ( n )th term in a geometric sequence?</td>
<td>1, 2</td>
<td>Page 31 in Lesson 1.3 (Example 1)</td>
</tr>
<tr>
<td>use a rule to determine any of ( n ), ( t_1 ), ( t_n ), or ( r ) in a geometric sequence?</td>
<td>3</td>
<td>Page 33 in Lesson 1.3 (Example 3)</td>
</tr>
<tr>
<td>model and solve problems involving geometric sequences?</td>
<td></td>
<td>Page 34 in Lesson 1.3 (Example 4)</td>
</tr>
<tr>
<td>use a rule to determine the sum ( S_n ) of a geometric series?</td>
<td>5</td>
<td>Page 45 in Lesson 1.4 (Example 1)</td>
</tr>
<tr>
<td>use a rule to determine ( t_1 ) in a geometric series, given the values of ( r ), ( n ), or ( S_n )?</td>
<td></td>
<td>Pages 45–46 in Lesson 1.4 (Example 2)</td>
</tr>
<tr>
<td>use a rule to determine ( n ) and ( S_n ) in a geometric series, given the values of ( t_1 ) and ( r )?</td>
<td>5</td>
<td>Page 46 in Lesson 1.4 (Example 3)</td>
</tr>
<tr>
<td>model and solve problems involving geometric series?</td>
<td>6</td>
<td>Page 47 in Lesson 1.4 (Example 4)</td>
</tr>
</tbody>
</table>

Assess Your Understanding

1.3

1. Multiple Choice For this geometric sequence: \(-5000, 500, -50, \ldots\); which number below is the value of \( t_9 \)?
   A. 0.0005  B. \(-0.0005\)  C. 0.000 05  D. \(-0.000 05\)

2. This sequence is geometric: 2, \(-6\), 18, \(-54\), \ldots
   a) Write a rule to determine the \( n \)th term.
      Use: \( t_n = t_1 r^{n-1} \)  Substitute: \( t_1 = 2 \), \( r = -3 \)
      \( t_n = 2(-3)^{n-1} \)
   b) Use your rule to determine the 10th term.
      Use: \( t_n = 2(-3)^{n-1} \)  Substitute: \( n = 10 \)
      \( t_{10} = 2(-3)^9 \)
      \( t_{10} = -39 366 \)
      The 10th term is \(-39 366\).
3. Use the given data about each geometric sequence to determine the indicated value.

a) \( t_4 = -5 \) and \( t_7 = 135 \); determine \( t_1 \)

Use \( t_n = t_1 r^{n-1} \) to determine \( r \).

First substitute: \( n = 4, t_4 = -5 \)

\(-5 = t_1 r^{4-1} \)

\(-5 = t_1 r^3 \) \( \Box \)

Then substitute: \( n = 7, t_7 = 135 \)

\( 135 = t_1 r^{7-1} \)

\( 135 = t_1 r^6 \) \( \Box \)

Write equation \( \Box \) as: \( 135 = t_1 r^3 (r^3) \)

From equation \( \Box \), substitute \( t_1 r^3 = -5 \)

\( 135 = -5 r^6 \) Divide by \(-5\).

\(-27 = r^6 \)

\( r = \sqrt[6]{-27} \)

\( r = -3 \)

Substitute \( r = -3 \) in equation \( \Box \).

\(-5 = t_1 (-3)^1 \)

\( t_1 = \frac{5}{-3} \)

b) \( t_1 = -1 \) and \( t_4 = -19,683 \); determine \( r \)

Use: \( t_n = t_1 r^{n-1} \)

Substitute: \( n = 4, t_4 = -19,683, t_1 = -1 \)

\(-19,683 = -1 r^{4-1} \)

\(-19,683 = -1 r^3 \)

\( 19,683 = r^4 \)

\( \sqrt[4]{19,683} = r \)

\( r = 27 \)

4. Multiple Choice The sum of the first 5 terms of a geometric series is \( \frac{121}{3} \). The common ratio is \( \frac{1}{3} \). What is the 2nd term?

A. \( \frac{1}{3} \) \quad B. 1 \quad C. \( \frac{121}{3} \) \quad D. \( \frac{1}{9} \)
5. Use the given data about each geometric series to determine the indicated value.

a) \( t_1 = -4, \ r = 3 \);
determine \( S_n \)

Use: \( S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \)
Substitute: \( n = 5, \ t_1 = -4, \ r = 3 \)
\[ S_5 = \frac{-4(1 - 3^5)}{1 - 3} \]
\[ S_5 = -484 \]

b) \( 3125 + 625 + 125 + \ldots + \frac{1}{25} \);
determine \( n \)

Use: \( t_n = t_1r^{n-1} \)
Substitute: \( t_n = \frac{1}{25}, \ t_1 = 3125, \ r = \frac{1}{5} \)
\[ 1 \]
\[ = 3125 \left( \frac{1}{5} \right)^{n-1} \]
\[ 1 \]
\[ = 78125 \left( \frac{1}{5} \right)^7 \]
\[ \left( \frac{1}{5} \right)^7 = \left( \frac{1}{5} \right)^{n-1} \]
\[ 7 = n - 1 \]
\[ n = 8 \]

6. The diagram shows a path of light reflected by mirrors.
After the first path, the length of each path is one-half the preceding length.

a) What is the length of the path from the 4th mirror to the 5th mirror?

The lengths of the paths, in centimetres, form this geometric sequence: 100, 50, 25, 12.5, 6.25, . . .
The length of the path from the 4th mirror to the 5th mirror is the 5th term: 6.25 cm

b) To the nearest hundredth of a centimetre, what is the total length of the path from the 1st mirror to the 10th mirror?

The total length of the path is the sum of the first 10 terms of this geometric series: 100 + 50 + 25 + 12.5 + 6.25 + . . .
Use: \( S_n = \frac{t_1(1 - r^n)}{1 - r} \), \( r \neq 1 \)
Substitute: \( n = 10, \ t_1 = 100, \ r = 0.5 \)
\[ S_{10} = \frac{100(1 - 0.5^{10})}{1 - 0.5} \]
\[ S_{10} = 199.8046 . . . \]
The path is approximately 199.80 cm long.

**Answers**

1. D 2. a) \( t_n = 2(-3)^{n-1} \) 3. a) \( \frac{5}{27} \) b) 27 4. A 5. a) -484  b) 8  6. a) 6.25 cm  b) 199.80 cm
Get Started

Here are 4 geometric sequences:

A. 1, 2, 4, 8, 16, . . .
B. 1, −2, 4, −8, 16, . . .
C. 1, 3/4, 3/8, 3/16, . . .
D. 1, −1/2, 1/4, −1/8, 1/16, . . .

Compare the sequences. How are they alike? How are they different?

All 4 sequences have 1st term 1. For Sequences A and B, the odd numbered terms are equal and the even numbered terms are opposites; so their common ratios are opposites. The same is true for Sequences C and D. The numbers in Sequence A are equal to the denominators in Sequence C, so their common ratios are reciprocals. The same is true for Sequences B and D.

Construct Understanding

Use a graphing calculator or graphing software to investigate graphs of geometric sequences and geometric series that have the same first term but different common ratios.

A. Choose a positive first term. Choose a common ratio, \( r \), in each of the intervals in the table below. For each common ratio, create the first 5 terms of a geometric sequence.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Common ratio, ( r )</th>
<th>Geometric sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r &gt; 1 )</td>
<td>( r = 2.5 )</td>
<td>100, 250, 625, 1562.5, 3906.25, . . .</td>
</tr>
<tr>
<td>( 0 &lt; r &lt; 1 )</td>
<td>( r = \frac{1}{4} )</td>
<td>100, 25, 6.25, 1.5625, 0.390625, . . .</td>
</tr>
<tr>
<td>( -1 &lt; r &lt; 0 )</td>
<td>( r = -0.2 )</td>
<td>100, −20, 4, −0.8, 0.16, . . .</td>
</tr>
<tr>
<td>( r &lt; -1 )</td>
<td>( r = -1.5 )</td>
<td>100, −150, 225, −337.5, 506.25, . . .</td>
</tr>
</tbody>
</table>
B. For each sequence

- Graph the term numbers on the horizontal axis and the term values on the vertical axis. Sketch and label each graph on a grid below, or print each graph.

**Sample response:**

![Graphs of geometric sequences]

- What happens to the term values as more points are plotted?

**As more points are plotted:**

For $r > 1$, the term values increase.

For $0 < r < 1$, the term values decrease and approach 0.

For $-1 < r < 0$, the term values alternate between positive and negative, and approach 0.

For $r < -1$, the term values alternate between positive and negative, and increase in numerical value.

**TEACHER NOTE**

Encourage students to use the terms *convergent* and *divergent* to describe the geometric sequences.

**Achievement Indicator**

*Construct Understanding,* Parts A to D, prepares students for a more formal treatment, in Lesson 1.6, related to AI 10.8: Explain why a geometric series is convergent or divergent.

**TEACHER NOTE**

*DI: Common Difficulties*

Some students may not see how the value of the common ratio affects the graphs of geometric sequences and series. Suggest a methodical approach: start with the same first term and create two pairs of examples, using common ratios of 2 and $-2$, 0.5 and $-0.5$. 

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C. Use the four geometric sequences in Part A to create four corresponding geometric series.

For each series

- Complete the table below by calculating these partial sums:
  \( S_1, S_2, S_3, S_4, S_5 \)

<table>
<thead>
<tr>
<th>Interval</th>
<th>Common ratio, ( r )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
<th>( S_4 )</th>
<th>( S_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r &gt; 1 )</td>
<td>( r = 2.5 )</td>
<td>100</td>
<td>350</td>
<td>975</td>
<td>2537.5</td>
<td>6443.75</td>
</tr>
<tr>
<td>( 0 &lt; r &lt; 1 )</td>
<td>( r = \frac{1}{4} )</td>
<td>100</td>
<td>125</td>
<td>131.25</td>
<td>132.8125</td>
<td>133.203</td>
</tr>
<tr>
<td>( -1 &lt; r &lt; 0 )</td>
<td>( r = -0.2 )</td>
<td>100</td>
<td>80</td>
<td>84</td>
<td>83.2</td>
<td>83.36</td>
</tr>
<tr>
<td>( r &lt; -1 )</td>
<td>( r = -1.5 )</td>
<td>100</td>
<td>-50</td>
<td>175</td>
<td>-162.5</td>
<td>343.75</td>
</tr>
</tbody>
</table>

- Graph the numbers of terms in the partial sums on the horizontal axis and the partial sums on the vertical axis. Sketch and label each graph on a grid below, or print each graph.

Sample response:

- Graph the numbers of terms in the partial sums on the horizontal axis and the partial sums on the vertical axis. Sketch and label each graph on a grid below, or print each graph.

Sample response:
• What happens to the partial sums as more points are plotted?

**As more points are plotted:**

For \( r > 1 \), the partial sums increase.

For \( 0 < r < 1 \), the partial sums increase and appear to approach a constant value close to 133.

For \( -1 < r < 0 \), the partial sums increase then decrease, and appear to approach a constant value close to 83.

For \( r < -1 \), the partial sums alternate between positive and negative values; the positive terms increase and the negative terms decrease.

**D. Without graphing**

• Describe the graph of this geometric sequence: 3, 2, \( \frac{4}{9} \), \( \frac{8}{27} \), ... 

The common ratio is \( \frac{2}{3} \), which is between 0 and 1, so the term values decrease and approach 0.

• Describe the graph of the partial sums of this geometric series:

\[ 3 + \frac{2}{3} + \frac{8}{9} + \frac{16}{27} + \ldots \]

The common ratio is \( \frac{2}{3} \), which is between 0 and 1, so the partial sums increase and approach a constant value.

Verify your descriptions by graphing. Sketch and label each graph on a grid below, or print each graph.

**Sample response:**

![Graphs of geometric sequence and series](image-url)
Assess Your Understanding

1. Create the first 5 terms of a geometric sequence with positive first term for each description of a graph.
   a) The term values approach 0 as more points are plotted.
      
      For the term values to approach 0, the common ratio must be between 0 and 1; for example, with \( r = 0.4 \), a possible sequence is:
      
      \( 2, 0.8, 0.32, 0.128, 0.0512, \ldots \)

   b) The term values increase as more points are plotted.
      
      For the term values to increase, the common ratio must be greater than 1; for example, with \( r = 4 \), a possible sequence is:
      
      \( 2, 8, 32, 128, 512, \ldots \)

   c) The term values alternate between positive and negative as more points are plotted.
      
      For the term values to alternate between positive and negative, the common ratio must be negative; for example, with \( r = -0.4 \), a possible sequence is:
      
      \( 2, -0.8, 0.32, -0.128, 0.0512, \ldots \)

2. Create a geometric series with positive first term for each description of a graph.
   a) The partial sums approach a constant value as more points are plotted.
      
      For the partial sums to approach a constant value, the common ratio must be between \( -1 \) and \( 1 \); for example, with \( r = 0.4 \), a possible series is:
      
      \( 2 + 0.8 + 0.32 + 0.128 + 0.0512 + \ldots \)

   b) The partial sums increase as more points are plotted.
      
      For the partial sums to increase, the common ratio must be greater than \( 1 \); for example, with \( r = 4 \), a possible series is:
      
      \( 2 + 8 + 32 + 128 + 512 + \ldots \)
1.6 Infinite Geometric Series

**FOCUS** Determine the sum of an infinite geometric series.

**Get Started**

Write $0.6$ as a series. What type of series is it? How do you know?

*Sample response:* $0.6 = 0.6666\ldots = 0.6 + 0.06 + 0.006 + 0.0006 + \ldots$

This is a geometric series because, after the 1st term, each term is $\frac{1}{10}$ of the preceding term.

**Construct Understanding**

Draw a square.
Divide it into 4 equal squares, then shade 1 smaller square.
Divide one smaller square into 4 equal squares, then shade 1 square.
Continue dividing smaller squares into 4 equal squares and shading 1 square, for as long as you can.
Suppose you could continue this process indefinitely.
Estimate the total area of the shaded squares. Explain your reasoning.

I created a sequence for the areas of the shaded squares. I wrote each partial sum, $S_n$, of the related series as a decimal.

<table>
<thead>
<tr>
<th>Square, $n$</th>
<th>Area of square</th>
<th>$S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{4}$</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{16}$</td>
<td>0.3125</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{64}$</td>
<td>0.32825</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{256}$</td>
<td>0.33203125</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{1024}$</td>
<td>0.3330078125</td>
</tr>
</tbody>
</table>

If I could continue the process indefinitely, the decimal that would represent the total area of the shaded squares would be $0.333\ldots$, or $\frac{1}{3}$.

So, I estimate that the total area of the shaded squares is $\frac{1}{3}$.
An infinite geometric series has an infinite number of terms. For an infinite geometric series, if the sequence of partial sums converges to a constant value as the number of terms increases, then the geometric series is convergent and the constant value is the finite sum of the series. This sum is called the sum to infinity and is denoted by $S_\infty$.

Example 1  
Estimating the Sum of an Infinite Geometric Series

Predict whether each infinite geometric series has a finite sum. Estimate each finite sum.

a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

b) $0.5 + 1 + 2 + 4 + \ldots$

c) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

SOLUTION

For each geometric series, calculate some partial sums.

a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$S_1 = \frac{1}{2}$  
$S_2 = S_1 + \frac{1}{4}$  
$S_3 = S_2 + \frac{1}{8}$  
$S_4 = S_3 + \frac{1}{16}$

$S_2 = \frac{1}{2} + \frac{1}{4}$  
$S_3 = \frac{3}{4} + \frac{1}{8}$  
$S_4 = \frac{7}{8} + \frac{1}{16}$

$S_3 = \frac{3}{4}$  
$S_4 = \frac{7}{8}$  
$S_4 = \frac{15}{16}$

The next term in the series is $\frac{1}{32}$.

b) $0.5 + 1 + 2 + 4 + \ldots$

$S_1 = 0.5$  
$S_2 = S_1 + 1$  
$S_3 = S_2 + 2$  
$S_4 = S_3 + 4$

$S_2 = 0.5 + 1$  
$S_3 = 1.5 + 2$  
$S_4 = 3.5 + 4$

$S_3 = 1.5$  
$S_4 = 3.5$  
$S_4 = 7.5$

As the number of terms increases, the partial sums increase, so the series does not have a finite sum.
In Example 1, each series has the same first term but different common ratios. So, from Example 1 and Lesson 1.5, it appears that the value of $r$ determines whether an infinite geometric series converges or diverges.

Consider the rule for the sum of $n$ terms.

For a geometric series, $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

When $-1 < r < 1$, $r^n$ approaches 0 as $n$ increases indefinitely.

So, $S_n$ approaches $\frac{t_1(1 - 0)}{1 - r}$

and, $S_{\infty} = \frac{t_1}{1 - r}$

THINK FURTHER

In Example 1, what other strategy could you use to estimate each finite sum?

I could graph the partial sums to see which constant value they approach.

In Example 1, why is each partial sum written as a decimal?

It is easier to identify the number to which the series appears to converge when the partial sums are written as decimals.

In Example 1, each series has the same first term but different common ratios. So, from Example 1 and Lesson 1.5, it appears that the value of $r$ determines whether an infinite geometric series converges or diverges.

Consider the rule for the sum of $n$ terms.

For a geometric series, $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

When $-1 < r < 1$, $r^n$ approaches 0 as $n$ increases indefinitely.

So, $S_n$ approaches $\frac{t_1(1 - 0)}{1 - r}$

and, $S_{\infty} = \frac{t_1}{1 - r}$

c) The next term in the series is $\frac{1}{100000}$.

$c) \quad S_1 = \frac{1}{10}, \text{ or } 0.1$

$c) \quad S_2 = S_1 - \frac{1}{100000}, \text{ or } 0.09$

$c) \quad S_3 = S_2 + \frac{1}{100000}, \text{ or } 0.091$

$c) \quad S_4 = S_3 - \frac{1}{100000}, \text{ or } 0.0909$

$c) \quad S_5 = S_4 + \frac{1}{100000}, \text{ or } 0.09091$

An estimate of the finite sum is 0.09.
Chapter 1: Sequences and Series

The Sum of an Infinite Geometric Series

For an infinite geometric series with first term, \( t_1 \), and common ratio, \( r \), the sum of the series, \( S \), is:

\[
S = \frac{t_1}{1 - r}
\]

When \( r \leq -1 \) or \( r \geq 1 \), the infinite geometric series diverges and does not have a finite sum.

Example 2

Determining the Sum of an Infinite Geometric Series

Determine whether each infinite geometric series converges or diverges. If it converges, determine its sum.

\[\text{a) } 27 - 9 + 3 - 1 + \ldots \quad \text{b) } 4 - 8 + 16 - 32 + \ldots\]

**SOLUTION**

**a)**

\( t_1 = 27 \) and \( r = \frac{-9}{27} = -\frac{1}{3} \)

The common ratio is between \(-1\) and \(1\), so the series converges. Use the rule for the sum of an infinite geometric series:

\[
S_\infty = \frac{t_1}{1 - r} = \frac{27}{1 - (-\frac{1}{3})} = \frac{27}{\frac{4}{3}} = 20.25
\]

The sum of the infinite geometric series is 20.25.

**b)**

\( t_1 = 4 \) and \( r = \frac{-8}{4} = -2 \)

The common ratio is not between \(-1\) and \(1\), so the series diverges. The infinite geometric series does not have a finite sum.

THINK FURTHER

In Example 2, how could you check that the sum in part a is reasonable?

I could determine a partial sum for a large number of terms such as \( S_{20} \). This sum should be very close to the value for \( S_\infty \).
Example 3  Using an Infinite Geometric Series to Solve a Problem

Determine a fraction that is equal to $0.4\overline{9}$.

**SOLUTION**

The repeating decimal $0.4\overline{9}$ can be expressed as:

$0.4 + 0.09 + 0.009 + 0.0009 + \ldots$

The repeating digits form an infinite geometric series. The series converges because $-1 < r < 1$. Use the rule for $S_\infty$.

\[
S_\infty = \frac{t_1}{1 - r}
\]

Substitute: $t_1 = 0.09$, or $\frac{9}{100}$; $r = 0.1$, or $\frac{1}{10}$

\[
S_\infty = \frac{\frac{9}{100}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1
\]

Add $\frac{1}{10}$ to 0.4, or $\frac{4}{10}$, the non-repeating part of the decimal:

\[
\frac{4}{10} + \frac{1}{10} = \frac{5}{10}, \text{ or } \frac{1}{2}
\]

So, $0.4\overline{9} = \frac{1}{2}$

### Check Your Understanding

3. Determine a fraction that is equal to $0.1\overline{6}$.

$0.1\overline{6} = 0.1 + 0.06 + 0.006 + 0.0006 + \ldots$

The repeating digits form an infinite geometric series with $t_1 = 0.06$, or $\frac{6}{100}$ and $r = 0.1$, or $\frac{1}{10}$

Use $S_\infty = \frac{t_1}{1 - r}$

Substitute for $t_1$ and $r$.

\[
S_\infty = \frac{\frac{6}{100}}{1 - \frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{9}, \text{ or } \frac{2}{30}
\]

Add $\frac{1}{10}$ to $\frac{2}{30}$, or $\frac{5}{30}$, or $\frac{1}{6}$

So, $0.1\overline{6} = \frac{1}{6}$

### Discuss the Ideas

1. How do you determine whether an infinite geometric series diverges or converges?

   I determine the common ratio, $r$. If $r$ is between $-1$ and $1$, the series converges; if $r \approx -1$ or $r \approx 1$, the series diverges.

2. An infinite geometric series has first term 5. Why does the series diverge when $r = 1$ or $r = -1$?

   When $r = 1$, all the terms of the series are equal and the partial sum will increase if the first term is positive and decrease if the first term is negative. When $r = -1$, the terms of the series have the same numerical value, but alternate in sign; so the partial sums alternate between a value equal to the first term and 0.

### TEACHER NOTE

**Achievement Indicator**

Question 1 addresses AI 10.8: Explain why a geometric series is convergent or divergent.
Exercises

3. Determine whether each infinite geometric series has a finite sum. How do you know?
   a) $2 + 3 + 4.5 + 6.75 + \ldots$
      \[
      r = \frac{3}{2} = 1.5, \text{ so the sum is not finite.}
      \]
   b) $-0.5 - 0.05 - 0.005 - 0.0005 - \ldots$
      \[
      r = \frac{-0.05}{-0.5} = 0.1, \text{ so the sum is finite.}
      \]
   c) $\frac{1}{2} - \frac{3}{8} + \frac{9}{32} - \frac{27}{128} + \ldots$
      \[
      r = \frac{-\frac{3}{2}}{\frac{1}{2}} = -\frac{3}{4}, \text{ so the sum is finite.}
      \]
   d) $0.1 + 0.2 + 0.4 + 0.8 + \ldots$
      \[
      r = \frac{0.2}{0.1} = 2, \text{ so the sum is not finite.}
      \]

4. Write the first 4 terms of each infinite geometric series.
   a) $t_1 = -4, r = 0.3$
      \[
      t_1 = -4, \\
      t_2 = -4(0.3) = -1.2, \\
      t_3 = -1.2(0.3) = -0.36, \\
      t_4 = -0.36(0.3) = -0.108
      \]
   b) $t_1 = 1, r = -0.25$
      \[
      t_1 = 1, \\
      t_2 = 1(-0.25) = -0.25, \\
      t_3 = -0.25(-0.25) = 0.0625, \\
      t_4 = 0.0625(-0.25) = -0.015625
      \]
   c) $t_1 = 4, r = \frac{1}{5}$
      \[
      t_1 = 4, \\
      t_2 = 4\left(\frac{1}{5}\right) = \frac{4}{5}, \\
      t_3 = \frac{4}{5}\left(\frac{1}{5}\right) = \frac{4}{25}, \\
      t_4 = \frac{4}{25}\left(\frac{1}{5}\right) = \frac{4}{125}
      \]
   d) $t_1 = -\frac{3}{2}, r = -\frac{3}{8}$
      \[
      t_1 = -\frac{3}{2}, \\
      t_2 = -\frac{3}{2}\left(-\frac{3}{8}\right) = \frac{9}{16}, \\
      t_3 = \frac{9}{16}\left(-\frac{3}{8}\right) = -\frac{27}{128}, \\
      t_4 = -\frac{27}{128}\left(-\frac{3}{8}\right) = \frac{81}{1024}
      \]
5. Each infinite geometric series converges. Determine each sum.

a) \(8 + 2 + 0.5 + 0.125 + \ldots\)
   
   Use: \(S_\infty = \frac{t_1}{1 - r}\)
   
   Substitute: \(t_1 = 8, r = \frac{1}{4}\)
   
   \(S_\infty = \frac{8}{1 - \frac{1}{4}}\)
   
   \(S_\infty = 10.6\)
   
   The sum is 10.6.

b) \(-1 - \frac{3}{4} - \frac{9}{16} - \frac{27}{64} - \ldots\)
   
   Use: \(S_\infty = \frac{t_1}{1 - r}\)
   
   Substitute: \(t_1 = -1, r = \frac{3}{4}\)
   
   \(S_\infty = \frac{-1}{1 - \frac{3}{4}}\)
   
   \(S_\infty = -4\)
   
   The sum is -4.

c) \(10 - \frac{20}{3} + \frac{40}{9} - \frac{80}{27} + \ldots\)
   
   Use: \(S_\infty = \frac{t_1}{1 - r}\)
   
   Substitute: \(t_1 = 10, r = -\frac{2}{3}\)
   
   \(S_\infty = \frac{10}{1 - (-\frac{2}{3})}\)
   
   \(S_\infty = 6\)
   
   The sum is 6.

d) \(-2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \ldots\)
   
   Use: \(S_\infty = \frac{t_1}{1 - r}\)
   
   Substitute: \(t_1 = -2, r = -\frac{1}{3}\)
   
   \(S_\infty = \frac{-2}{1 - (-\frac{1}{3})}\)
   
   \(S_\infty = -1.5\)
   
   The sum is -1.5.

6. What do you know about the common ratio of an infinite geometric series whose sum is finite?

   The common ratio is less than 1 and greater than -1.

7. Use the given data about each infinite geometric series to determine the indicated value.

   a) \(t_1 = 21, S_\infty = 63;\) determine \(r\)
   
   Substitute for \(t_1\) and \(S_\infty\)
   
   in \(S_\infty = \frac{t_1}{1 - r}\)
   
   \(63 = \frac{21}{1 - r}\)
   
   \(63 - 63r = 21\)
   
   \(63r = 42\)
   
   \(r = \frac{42}{63} = \frac{2}{3}\)

   b) \(r = -\frac{3}{4}, S_\infty = \frac{24}{7};\) determine \(t_1\)
   
   Substitute for \(r\) and \(S_\infty\)
   
   in \(S_\infty = \frac{t_1}{1 - r}\)
   
   \(\frac{24}{7} = \frac{t_1}{1 - (-\frac{3}{4})}\)
   
   \(t_1 = 6\)
8. Use an infinite geometric series to express each repeating decimal as a fraction.

a) \(0.4\overline{97}\)

\[
0.4\overline{97} = 0.4 + \frac{97}{1000} + \frac{97}{10000} + \ldots
\]

This is an infinite geometric series with \(t_1 = \frac{97}{1000}\) and \(r = \frac{97}{100}\). Substitute for \(t_1\) and \(r\) in \(S_\infty = \frac{t_1}{1 - r}\):

\[
S_\infty = \frac{\frac{97}{1000}}{1 - \frac{97}{100}} = \frac{97}{990}
\]

Add: \(\frac{4}{10} + \frac{97}{990} = \frac{93}{990}\)

So, \(0.4\overline{97} = \frac{93}{990}\)

b) \(1.\overline{143}\)

\[
1.\overline{143} = 1 + \frac{143}{1000} + \frac{143}{10000} + \ldots
\]

This is an infinite geometric series with \(t_1 = \frac{143}{1000}\) and \(r = \frac{1}{10}\). Substitute for \(t_1\) and \(r\) in \(S_\infty = \frac{t_1}{1 - r}\):

\[
S_\infty = \frac{\frac{143}{1000}}{1 - \frac{1}{10}} = \frac{143}{999}
\]

Add: \(1 + \frac{143}{999} = \frac{1142}{999}\)

So, \(1.\overline{143} = \frac{1142}{999}\)

9. The hour hand on a clock is pointing to 12. The hand is rotated clockwise 180°, then another 60°, then another 20°, and so on. This pattern continues.

a) Which number would the hour hand approach if this rotation continued indefinitely? Explain what you did.

The angles, in degrees, that the hand rotates through form a geometric sequence with \(t_1 = 180\) and \(r = \frac{1}{3}\). The total angle turned through is the related infinite geometric series:

\[
180 + \frac{180}{3} + \frac{180}{3^2} + \frac{180}{3^3} + \ldots
\]

Use: \(S_\infty = \frac{t_1}{1 - r}\)

Substitute: \(t_1 = 180, r = \frac{1}{3}\)

\[
S_\infty = \frac{180}{1 - \frac{1}{3}} = \frac{270}{2}
\]

When the hour hand has rotated 270° clockwise from 12, it will point to 9. So, if the rotation continued indefinitely, the hour hand would approach 9.

b) What assumptions did you make?

I assumed that the angle measures formed an infinite geometric series that converged.

TEACHER NOTE

Di: Common Difficulties

For students having difficulty expressing a repeating decimal as an infinite geometric series, suggest they write 3 or more repetitions of the repeating period, then record the decimal expansion in a place-value chart. From there, they can use place value (powers of 10) to identify the terms in the infinite geometric series.

Achievement Indicator

Questions 9, 10, and 11 address AI 10.9:

Solve a problem that involves a geometric series.
10. Brad has a balance of $500 in a bank account. Each month he spends 40% of the balance remaining in the account.

a) Express the total amount Brad spends in the first 4 months as a series. Is the series geometric? Explain.

<table>
<thead>
<tr>
<th>After:</th>
<th>Amount spent</th>
<th>Amount remaining</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>$500(0.4) = $200</td>
<td>$500(0.6) = $300</td>
</tr>
<tr>
<td>2 months</td>
<td>$300(0.4) = $120,</td>
<td>$300(0.6) = $180,</td>
</tr>
<tr>
<td></td>
<td>or $500(0.6)(0.4)</td>
<td>or $500(0.6)^2</td>
</tr>
<tr>
<td>3 months</td>
<td>$180(0.4) = $72,</td>
<td>$180(0.6) = $108,</td>
</tr>
<tr>
<td></td>
<td>or $500(0.6)^2(0.4)</td>
<td>or $500(0.6)^3</td>
</tr>
<tr>
<td>4 months</td>
<td>$108(0.4) = $43.20,</td>
<td>or $500(0.6)^3(0.4)</td>
</tr>
</tbody>
</table>

The amounts spent are:

$500(0.4) + 500(0.6)(0.4) + 500(0.6)^2(0.4) + 500(0.6)^3(0.4)

This is a geometric series with \( t_1 = 500(0.4) \) and \( r = 0.6 \)

b) Determine the approximate amount Brad spends in 10 months. Explain what you did.

The amount Brad spends in 10 months is the sum of the first 10 terms of the series in part a.

\[ S_{10} = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1 \]

Substitute: \( n = 10, t_1 = 200, r = 0.6 \)

\[ S_{10} = \frac{200(1 - 0.6^{10})}{1 - 0.6} \]

\[ S_{10} = 496.9766... \]

Brad spends about $496.98 in 10 months.

c) Suppose Brad could continue this pattern of spending indefinitely. Would he eventually empty his bank account? Explain.

No, because Brad can only spend money in dollars and cents, and not fractions of a cent, so each amount he spends will be rounded to the nearest cent. Continuing the pattern of spending 40% each month, and rounding to the nearest cent each time, Brad will eventually end up with $0.01 remaining in his account. Since 40% of $0.01 is less than 1 penny, this amount will never be spent.
11. Write the product of $0.a$ and $0.b$ as a fraction, where $a$ and $b$ represent 1-digit natural numbers. Explain your strategy.

\[ 0.a = 0.a + 0.0a + 0.00a + \ldots \]

This is an infinite geometric series with $t_1 = 0.a$, or $\frac{a}{10}$ and $r = 0.1$, or $\frac{1}{10}$.

Use: $S_\infty = \frac{t_1}{1 - r}$
Substitute: $t_1 = \frac{a}{10}$, $r = \frac{1}{10}$

\[ S_\infty = \frac{a}{10} \cdot \frac{1 - \left(\frac{1}{10}\right)}{1} = \frac{a}{10} \cdot \frac{9}{10} = \frac{9a}{100} \]

So, $0.a = \frac{a}{9}$; similarly, $0.b = \frac{b}{9}$; then $(0.a)(0.b)$ is

\[ \left(\frac{a}{9}\right) \left(\frac{b}{9}\right) = \frac{ab}{81} \]


Sample response: Choose a value for $r$ between $-1$ and $1$, then determine a value for $t_1$.

Let $r = -0.25$.

Use: $S_\infty = \frac{t_1}{1 - r}$
Substitute:
$S_\infty = 4$, $r = -0.25$

\[ 4 = \frac{t_1}{1 - (-0.25)} \]

$t_1 = 5$

One series is: $5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots$

Another series is: $1.6 + 0.96 + 0.576 + 0.3456 + \ldots$
13. Determine the sum of this infinite geometric series:
\[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{32}} + \frac{1}{\sqrt{128}} + \frac{1}{\sqrt{512}} + \ldots \]

The common ratio is \( r = \frac{\sqrt{2}}{2} \), or \( \frac{1}{2} \).

Use: \( S_\infty = \frac{t_1}{1-r} \) Substituting: \( t_1 = \frac{1}{\sqrt{2}} \) \( r = \frac{1}{2} \)

\[ S_\infty = \frac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{2}} \]

The sum is \( \frac{2}{\sqrt{2}} \).

**Multiple-Choice Questions**

1. What is the sum of this infinite geometric series?
\[ 10 - \frac{20}{3} + \frac{40}{9} - \frac{80}{27} + \ldots \]
- A. 30
- B. 4
- C. 6
- D. 20

2. Which infinite geometric series has the sum \(-8.3\)?
- A. \(-5 - 2 - 0.8 - 0.32 - \ldots \)
- B. \(-5 + 2 - 0.8 + 0.32 - \ldots \)
- C. \(5 - 2 + 0.8 - 0.32 + \ldots \)
- D. \(5 + 2 + 0.8 + 0.32 + \ldots \)

3. How many of these geometric series have finite sums?
\[ 1 + 0.5 + 0.125 + 0.0625 + \ldots \]
\[ 1 + \frac{4}{3} + \frac{16}{27} + \frac{64}{27} + \ldots \]
- A. 1 series
- B. 2 series
- C. 3 series
- D. 4 series

**Study Note**

What is a rule for determining the sum of an infinite geometric series? When is it appropriate to apply this rule? When is it not appropriate?

The rule for the sum of an infinite geometric series, \( S_\infty \), is \( S_\infty = \frac{t_1}{1-r} \), where \( t_1 \) is the first term and \( r \) is the common ratio. This rule may be applied when \( r \) is between \(-1 \) and \( 1 \). The rule may not be applied when \( r \leq -1 \) or \( r \geq 1 \).

**ANSWERS**

3. a) not finite b) finite c) finite d) not finite
4. a) \(-4 - 1.2 - 0.36 - 0.108\)
  b) \(1 - 0.25 + 0.0625 - 0.015625\)
  c) \(4 + \frac{4}{5} + \frac{4}{25} + \frac{4}{125}\)
  d) \(-\frac{3}{2} + \frac{9}{16} - \frac{27}{128} + \frac{81}{1024}\)
5. a) 10.5 b) 4 c) 6 d) -1.5 e) \(2\frac{2}{5}\)
6. a) \(\frac{493}{999}\) b) \(\frac{1142}{999}\)
7. a) 9
8. a) \(\frac{29}{999}\)
9. a) 9
10. a) \$500(0.4) + \$500(0.6)(0.4) + \$500(0.6)^2(0.4) + \$500(0.6)^3(0.4)\); geometric
  b) \$496.98
  c) yes

**Multiple Choice**
# STUDY GUIDE

## Concept Summary

<table>
<thead>
<tr>
<th>Big Ideas</th>
<th>Applying the Big Ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>• An arithmetic sequence is related to a linear function and is created by repeatedly adding a constant to an initial number. An arithmetic series is the sum of the terms of an arithmetic sequence.</td>
<td>This means that:</td>
</tr>
<tr>
<td></td>
<td>• The common difference of an arithmetic sequence is equal to the slope of the line through the points of the graph of the related linear function.</td>
</tr>
<tr>
<td></td>
<td>• Rules can be derived to determine the nth term of an arithmetic sequence and the sum of the first n terms of an arithmetic series.</td>
</tr>
<tr>
<td>• A geometric sequence is created by repeatedly multiplying an initial number by a constant. A geometric series is the sum of the terms of a geometric sequence.</td>
<td>• The common ratio of a geometric sequence can be determined by dividing any term after the first term by the preceding term.</td>
</tr>
<tr>
<td></td>
<td>• Rules can be derived to determine the nth term of a geometric sequence and the sum of the first n terms of a geometric series.</td>
</tr>
<tr>
<td>• Any finite series has a sum, but an infinite geometric series may or may not have a sum.</td>
<td>• The common ratio determines whether an infinite series has a finite sum.</td>
</tr>
</tbody>
</table>

## Chapter Study Notes

- What information do you need to know about an arithmetic sequence and a geometric sequence to determine $t_n$?

  **For an arithmetic sequence, I need to know the first term, $t_1$, and the common difference, $d$. Then I can use the rule:**
  
  $t_n = t_1 + d(n - 1)$

  **For a geometric sequence, I need to know the first term, $t_1$, and the common ratio, $r$. Then I can use the rule:**
  
  $t_n = t_1 r^{n-1}$

- What information do you need to know about an arithmetic series and a geometric series to determine the sum $S_n$?

  **For an arithmetic series, I need to know the first term, $t_1$, the common difference, $d$, and the number of terms, $n$. Then I can use the rule:**
  
  $S_n = \frac{n(t_1 + t_n)}{2}$

  Alternatively, for an arithmetic series, I need to know the first term, $t_1$, the number of terms, $n$, and the last term, $t_n$. Then I can use the rule:
  
  $S_n = \frac{n(t_1 + t_n)}{2}$

  **For a geometric series, I need to know the first term, $t_1$, the common ratio, $r$, and the number of terms, $n$. Then I can use the rule:**
  
  $S_n = \frac{t_1(1 - r^n)}{1 - r}, \quad r \neq 1$
### Skills Summary

<table>
<thead>
<tr>
<th>Skill</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the general term, (t_n), for an arithmetic sequence. (1.1, 1.2)</td>
<td>A rule is: (t_n = t_1 + d(n - 1)) where (t_1) is the first term, (d) is the common difference, and (n) is the number of terms.</td>
<td>For this arithmetic sequence: (-9, -3, 3, 9, \ldots) the 20th term is: (t_{20} = -9 + 6(20 - 1)) (t_{20} = -9 + 6(19)) (t_{20} = 105)</td>
</tr>
<tr>
<td>Determine the sum of (n) terms, (S_n), for an arithmetic series. (1.2)</td>
<td>When (n) is the number of terms, (t_1) is the first term, (t_n) is the (n)th term, and (d) is the common difference One rule is: (S_n = \frac{n(t_1 + t_n)}{2}) Another rule is: (S_n = \frac{n[2t_1 + d(n - 1)]}{2})</td>
<td>For this arithmetic series: (5 + 7 + 9 + 11 + 13 + 15 + 17); the sum of the first 7 terms is: (S_7 = \frac{7(5 + 17)}{2}) (S_7 = \frac{7(22)}{2}) (S_7 = 77)</td>
</tr>
<tr>
<td>Determine the general term, (t_n), for a geometric sequence. (1.3, 1.4)</td>
<td>A rule is: (t_n = t_1r^{n-1}) where (t_1) is the first term, (r) is the common ratio, and (n) is the number of terms.</td>
<td>For this geometric sequence: (1, -0.25, 0.0625, \ldots) the 6th term is: (t_6 = (-0.25)^{6-1}) (t_6 = (-0.25)^5) (t_6 = -0.0009765)</td>
</tr>
<tr>
<td>Determine the sum of (n) terms, (S_n), of a geometric series. (1.4)</td>
<td>A rule is: (S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1) where (t_1) is the first term, (r) is the common ratio, and (n) is the number of terms.</td>
<td>For this geometric series: (4, 2, 1, \ldots) the sum of the first 10 terms is: (S_{10} = \frac{4(1 - 0.5^{10})}{1 - 0.5}) (S_{10} = 7.9921\ldots) or approximately 8</td>
</tr>
<tr>
<td>Determine the sum, (S), of a convergent infinite geometric series. (1.6)</td>
<td>When (r) is between (-1) and 1, use this rule: (S_n = \frac{t_1}{1 - r}) where (t_1) is the first term and (r) is the common ratio.</td>
<td>For this geometric series: (100 - 50 + 25 - \ldots) the sum is: (S_n = \frac{100}{1 - (-0.5)}) (S_n = 66.6)</td>
</tr>
</tbody>
</table>
1.1

During the 2003 fire season, the Okanagan Mountain Park fire was the most significant wildfire event in B.C. history. By September 7, the area burned had reached about 24,900 ha and the fire was spreading at a rate of about 150 ha/h.

a) Suppose the fire continued to spread at the same rate. Create terms of a sequence to represent the area burned for each of the next 6 h. Why is the sequence arithmetic?

Each hour, the area increases by 150 ha. So, for each of the next 6 h, the area burned in hectares is: 25,050, 25,200, 25,350, 25,500, 25,650, 25,800, . . . This sequence is arithmetic because the difference between consecutive terms is constant.

b) Write a rule for the general term of the sequence in part a. Use the rule to predict the area burned after 24 h. What assumptions did you make?

Use: \( t_n = t_1 + d(n - 1) \) Substitute: \( t_1 = 25,050, d = 150 \n\)
The general term is: \( t_n = 25,050 + 150(n - 1) \n\)
Substitute: \( n = 24 \n\)
\( t_{24} = 25,050 + 150(24 - 1) \n\)
\( t_{24} = 28,500 \n\)
After 24 h, the area burned was 28,500 ha; I assumed that the fire continued to spread at the same rate.

1.2

Use the given data about each arithmetic series to determine the indicated value.

a) \( 5 + \frac{3}{2} + 2 + \frac{1}{2} - 1 - \ldots \); determine \( S_{21} \)

Use: \( S_n = \frac{n[2t_1 + d(n - 1)]}{2} \n\)
Substitute: \( n = 21, t_1 = 5, d = -1.5 \n\)
\( S_{21} = \frac{21[2(5) - 1.5(21 - 1)]}{2} \n\)
\( S_{21} = -210 \n\)

b) \( S_{12} = 78 \) and \( t_1 = -21 \); determine \( t_{12} \)

Use: \( S_n = \frac{n(t_1 + t_n)}{2} \n\)
Substitute: \( n = 12, S_{12} = 78, t_1 = -21 \n\)
\( 78 = \frac{12(-21 + t_{12})}{2} \n\)
\( 78 = 6(-21 + t_{12}) \n\)
\( 13 = -21 + t_{12} \n\)
\( t_{12} = 34 \n\)
3. Explain the meaning of this newspaper headline.

I-Pod Sales Grew Geometrically from 2001 to 2006

In 2002, the number of sales was equal to a constant multiplied by the number of sales in 2001. In 2003, the number of sales was equal to the same constant multiplied by the number of sales in 2002. This pattern continued up to 2006.

4. A soapstone carving was appraised at $2500. The value of the carving is estimated to increase by 12% each year. What will be the approximate value of the carving after 15 years?

The values of the carving, in dollars, form a geometric sequence with \( t_1 = 2500 \) and \( r = 1.12 \). The value, in dollars, after 15 years is \( t_{16} \).

Use \( t_n = t_1 r^{n-1} \) to determine \( n \).

Substitute: \( t_n = 5.46875, \) \( t_1 = -700, \) \( r = -0.5 \):

\[
5.46875 = -700(-0.5)^{n-1} \\
-0.0078125 = (-0.5)^{n-1} \\
(-0.5)^1 = (-0.5)^{n-1} \\
\therefore \quad n = 8
\]

Then, use: \( S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \).

Substitute: \( n = 8, \) \( t_1 = -700, \) \( r = -0.5 \):

\[
S_8 = \frac{-700[1 - (-0.5)^8]}{1 - (-0.5)} \\
\approx -464.844
\]

1.3

5. Determine the sum of the geometric series below. Give the answer to 3 decimal places.

\[-700 + 350 - 175 + \ldots + 5.46875\]

Use: \( t_n = t_1 r^{n-1} \) to determine \( n \).

Substitute: \( t_n = 5.46875, \ t_1 = -700, \ r = 0.5 \):

\[
5.46875 = -700(0.5)^{n-1} \\
-0.0078125 = (0.5)^{n-1} \\
(0.5)^1 = (0.5)^{n-1} \\
\therefore \quad n = 8
\]

Then, use: \( S_n = \frac{t_1(1 - r^n)}{1 - r}, \ r \neq 1 \).

Substitute: \( n = 8, \ t_1 = -700, \ r = -0.5 \):

\[
S_8 = \frac{-700[1 - (-0.5)^8]}{1 - (-0.5)} \\
\approx -464.844
\]
6. Use a graphing calculator or graphing software.
   Use the series from question 5. Graph the first 5 partial sums.
   Explain how the graph shows whether the series converges or diverges.

   Sample response: The series has \( t_1 = -700 \) and \( r = -0.5 \); its partial sums are: 
   \(-700, -350, -175, -87.5, -43.75, \ldots \)
   The series converges because the points appear to approach a constant value of approximately \(-450\).

7. Explain how you can use the common ratio of a geometric series to identify whether the series is convergent or divergent.

   A geometric series with a common ratio less than 1 and greater than \(-1\) converges. A geometric series with a common ratio less than or equal to \(-1\) or greater than or equal to 1 diverges.

8. Identify each infinite geometric series that converges. Determine the sum of any series that converges.
   a) \( 2 - 3 + 4.5 - 6.75 + \ldots \)  
      \( r = \frac{-3}{2} = -1.5 \),  
      so the series diverges.
   b) \( \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \ldots \)  
      \( r = \frac{2}{3} \)  
      so the series converges.

   Use: \( S_\infty = \frac{t_1}{1 - r} \)

   Substitute: \( t_1 = \frac{1}{3}, r = \frac{2}{3} \)

   \( S_\infty = \frac{\frac{1}{3}}{1 - \frac{2}{3}}, \) or 1
9. A small steel ball bearing is moving vertically between two electromagnets whose relative strength varies each second. The ball bearing moves 10 cm up in the 1st second, then 5 cm down in the 2nd second, then 2.5 cm up in 3rd second, and so on. This pattern continues.

a) Assume the distance the ball bearing moves up is positive; the distance it moves down is negative.

i) Write a series to represent the distance travelled in 5 s.

Each second, the distance is halved.
In the first 5 s, the distance in centimetres is:
10 – 5 + 2.5 – 1.25 + 0.625

ii) Calculate the sum of the series. What does this sum represent?

Use: \( S_n = \frac{t_1(1 - r^n)}{1 - r} \), Substitute: \( n = 5, t_1 = 10, r = -0.5 \)
\[ S_5 = \frac{10(1 - (-0.5)^5)}{1 - (-0.5)} \]
\[ S_5 = 6.875 \]
After 5 s, the ball bearing is 6.875 cm above the magnet from which it started.

b) Suppose this process continues indefinitely. What is the sum of the series?

The series is infinite and converges.

Use: \( S_\infty = \frac{t_1}{1 - r} \) Substitute: \( t_1 = 10, r = -0.5 \)
\( S_\infty = \frac{10}{1 - (-0.5)} \)
\[ S_\infty = 6.6 \]
The sum of the series is 6.6 cm.

ANSWERS
1. a) 25 050, 25 200, 25 350, 25 500, 25 650, 25 800, ...   b) approximately 28 500 ha
2. a) –210   b) 34   c) 5.8  6. 4  8. a) diverges  
   b) converges; 1  9. a) i) 10 – 5 + 2.5 – 1.25 + 0.625  
   ii) 6.875 cm  b) 6.6 cm
1. Multiple Choice What is the sum of the first 30 terms of this arithmetic series? \(-5 - 2 + 1 + 4 + \ldots\)
   A. 1152  B. 1155  C. 1158  D. 1161

2. Multiple Choice What is the sum of the first 10 terms of this geometric series?
   \(-12800 + 6400 - 3200 + 1600 - \ldots\)
   A. 8525  B. 8537.5  C. -8537.5  D. 8537.5

3. a) Which sequence below appears to be arithmetic? Justify your answer.
   i) 4, -10, 16, -22, 28, \ldots  ii) 4, -10, -24, -38, -52, \ldots
   In part i, the differences of consecutive terms are: 
   \(-14, 26, -38, 50\). Since these differences are not equal, the sequence is not arithmetic.
   In part ii, the differences of consecutive terms are: 
   \(-14, -14, -14, -14\). Since these differences are equal, the sequence appears to be arithmetic.

b) Assume that the sequence you identified in part a is arithmetic. Determine:
   i) a rule for \(t_n\)  ii) \(t_{17}\)
   The arithmetic sequence is: \(4, -10, -24, -38, -52, \ldots\)
   Use: \(t_n = t_1 + d(n - 1)\)
   Substitute: \(n = 17\)
   \(t_{17} = 4 - 14(17 - 1)\)
   Substitute: \(t_1 = 4, d = -14\)
   \(t_{17} = -220\)
   \(t_n = 4 - 14(n - 1)\)

iii) the term that has value \(-332\)
   Use: \(t_n = 4 - 14(n - 1)\)  Substitute: \(t_n = -332\)
   \(-332 = 4 - 14(n - 1)\)
   \(24 = n - 1\)
   \(n = 25\)
   The 25th term has value \(-332\).
4. For a geometric sequence, \( t_4 = -1000 \) and \( t_7 = 1 \); determine:
   a) \( t_1 \)
   b) the term with value 0.0001

   a) \( t_1 \)
   - Use: \( t_n = t_1 r^{n-1} \)
   - Substitute:
     \[ t_7 = 1, t_4 = -1000 \]
     \[ r^3 = \frac{-1}{1000} \]
     \[ r = \frac{-1}{10} \text{ or } -0.1 \]
   - Use: \( t_n = t_1 r^{n-1} \)
   - Substitute:
     \[ n = 7, t_7 = 1, r = -0.1 \]
     \[ t_1 = \frac{1}{(-0.1)^7} \]
     \[ t_1 = 1000000 \]

   b) the term with value 0.0001
   - Use: \( t_n = t_1 r^{n-1} \)
   - Substitute:
     \[ t_6 = 0.0001, t_1 = 1000000, \]
     \[ r = -0.1 \]
     \[ 0.0001 = 1000000(-0.1)^{n-1} \]
     \[ n = 11 \]
   - The 11th term has value 0.0001.

5. a) For the infinite geometric series below, identify which series converges and which series diverges. Justify your answer.
   i) \( 100 - 150 + 225 - 337.5 + \ldots \)
   - The common ratio, \( r \), is: \( \frac{-150}{100} = -1.5 \)
   - Since \( r \) is less than \(-1\), the series diverges.

   ii) \( 10 + 5 + 2.5 + 1.25 + \ldots \)
   - \( r \) is: \( \frac{5}{10} = \frac{1}{2} \)
   - Since \( r \) is between \(-1\) and 1, the series converges.

b) For which series in part a can you determine its sum? Explain why, then determine this sum.
   I can determine the sum of an infinite geometric series that converges; that is, the series in part a ii).
   - Use: \( S_\infty = \frac{t_1}{1 - r} \)
   - Substitute: \( t_1 = 10, r = \frac{1}{2}, \text{ or } 0.5 \)
   - \( S_\infty = \frac{10}{1 - 0.5}, \text{ or } 20 \)
   - The sum of the series is 20.

TEACHER NOTE

Achievement Indicators

Question 4 addresses AI 10.6:
Determine \( t_1, r, n, \) or \( S_n \) in a problem that involves a geometric series.

Question 5 addresses
AI 10.6: Determine \( t_1, r, n, \) or \( S_n \) in a problem that involves a geometric series.
AI 10.7: Generalize, using inductive reasoning, a rule for determining the sum of an infinite geometric series.
AI 10.8: Explain why a geometric series is convergent or divergent.
6. This sequence represents the approximate lengths in centimetres of a spring that is stretched by loading it with from one to four 5-kg masses: 50, 54, 58, 62, . . .
Suppose the pattern in the sequence continues. What will the length of the spring be when it is loaded with ten 5-kg masses? Explain how you found out.

Since the differences between consecutive terms are equal, then the series appears to be arithmetic. The length of the spring, in centimetres, will be the 10th term of the arithmetic sequence.

Use: \( t_n = t_1 + d(n - 1) \)  
Substitute: \( n = 10, t_1 = 50, d = 4 \)

\[ t_n = 50 + 4(10 - 1) \]

\[ t_n = 86 \]

The spring will be 86 cm long.

7. As part of his exercise routine, Earl uses a program designed to help him eventually do 100 consecutive push-ups. He started with 17 push-ups in week 1 and planned to increase the number of push-ups by 2 each week.

a) In which week does Earl expect to reach his goal?

The number of push-ups each week form an arithmetic sequence with \( t_1 = 17 \) and \( d = 2 \). Determine \( n \) for \( t_n = 100 \).

Use: \( t_n = t_1 + d(n - 1) \)  
Substitute: \( t_n = 100, t_1 = 17, d = 2 \)

\[ 100 = 17 + 2(n - 1) \]

\[ 83 = 2n - 2 \]

\[ 2n = 85 \]

\[ n = 42.5 \]

Earl should reach his goal in the 43rd week.

b) What is the total number of push-ups he will have done when he reaches his goal? Explain how you know.

The total number of push-ups is the sum of the first 43 terms of the arithmetic sequence.

Use: \( S_n = \frac{n(t_1 + t_n)}{2} \)  
Substitute: \( n = 43, t_1 = 17, d = 2 \)

\[ S_{43} = \frac{43(17 + 2(43 - 1))}{2} \]

\[ S_{43} = 2537 \]

Earl will have done 2537 push-ups when he reaches his goal.