Supporting Materials

CME Project ©2013 Algebra 1, Geometry, Algebra 2, PreCalculus Common Core Edition includes a wealth of teaching resources that make teaching easier and support the transition from a state standards-based curriculum to one that embraces all the concepts and skills that comprise the Common Core State Standards. The program features a rich array of diagnostic, formative, and summative assessment tools so that you can assess and remediate your students’ progress every step of the way.

Inside this sampler you will find:

• Overview of all program components

• Example of teaching resources
  • Additional Practice Worksheets
  • Lesson Quizzes
  • Chapter Test

• CME Implementation Guide
Common Core Student Edition
This hardbound text contains 8 chapters. Each chapter contains 3-4 Investigations. Each Investigation contains 3-6 Lessons. Each Chapter also includes a Mid-Chapter Test, Chapter Review, Chapter Test, and Project. A TI-Nspire™ Appendix is also included to support students’ use of this powerful new technology. Also available as an eText on Pearson SuccessNet and for iPad/Android tablets.

Common Core Teacher’s Edition
The structure of the CME Project Teacher’s Edition provides the support you need at point-of-use.
- Detailed math background for every chapter and lesson
- Complete support for pacing and resource management
- Detailed, daily teaching plans
- Support for differentiating instruction
- Assignment guide
- Also available as an eText on Pearson SuccessNet and for iPad/Android tablets

Implementation Guide (grades 9-12 resource)
This powerful teacher resource provides a detailed overview of the CME Project along with specific recommendations for teachers and districts on implementing the program.

Additional Practice Workbook (available one per grade level)
This student consumable provides additional practice for every lesson.

Assessment Resources
All your program assessments, including lesson quizzes & mid-chapter, chapter, quarter, and mid-year tests are located in one convenient location on Pearson SuccessNet.
Teaching Resources
Answers to the Additional Practice Workbook as well as blackline masters of resources that can be used as transparencies or handouts are included in this resource to support teachers during instruction.

Solutions Manual
Worked-out solutions for every exercise.

ExamView® Test Generator CD-ROM (grades 9-12 resource)
Choose from an extensive question bank, containing questions written for the CME Project, to quickly and easily create level practice worksheets and test. Correlations to state standards are available so users can progress monitor student achievement on essential content for their state.

Pearson SuccessNet
Pearson SuccessNet is a complete K–12, online teaching and learning environment. It’s the one site where everyone—teachers, students, parents, and administrators—can access program resources, such as:
- Student and Teacher Edition eText
- Teaching and Assessment Resources
- Homework Video Tutors
- Online Practice Activities

SuccessTracker™ (available per grade level)
Personalized intervention for each student. Assess, diagnose, remediate, and report student performance throughout the year so there are no “surprises” come the “big test.”
1. Messenger Service A charges $15 for pick up and $1.50 per mile for delivery. Messenger Service B charges $9 for pick up and $2.00 per mile for delivery.
   a. If the delivery distance is 10 miles, which company charges less? How much less?
   b. If the delivery distance is 25 miles, which company charges less? How much less?
   c. At what point will both companies charge the same amount?

2. Use substitution to find the intersection point of the graphs of the system of equations.
   a. \( y = 6 - 2x \) and \( 4x + 2y = 8 \)
   b. Graph each equation. What do the graphs suggest about the solution to the system of equations?

3. What is the solution to each system of equations? If there is no solution, explain.
   a. \( y = 19x + 92 \) and \( y = 25x - 40 \)
   b. \( y = 23x + 175 \) and \( y = 23x - 150 \)

4. Identify the graphs of each pair of equations as parallel, intersecting, or identical. Explain.
   a. \( y = \frac{2}{5}x + 7 \) and \( y = 0.4x - 1 \)
   b. \( 3x - 2y = 1 \) and \( 2x - 3y = -1 \)
   c. \( 3x + 2y = 5 \) and \( -3x - 2y = -5 \)
   d. \( 2x - 3y = 6 \) and \( y = \frac{2}{3}x - 6 \)

5. Write an equation of the line that contains the given point and is parallel to the given line.
   a. \((12, -3); y = -\frac{5}{6}x - 3\)
   b. \((-6, -5); 2x - 3y = 12\)
   c. \((1, -4); y - 2 = -\frac{2}{5}(x + 11)\)
   d. \((-2, -11); x = -12\frac{2}{5}\)
   e. \((-2, -11); y = 15.75\)

6. What is the value of \( k \) such that the graph of the equation will pass through \((0, -8)\)?
   a. \( y = 5x + k \)
   b. \( y = 6x + k \)
   c. \( y = 7x + k \)
   d. \( y = 100x + k \)
   e. \( y = qx + k \)
   f. What pattern do you notice about the values of \( k \)?

7. What is the value of \( k \) such that the graph of the equation will pass through \((-8, 0)\)?
   a. \( y = 5x + k \)
   b. \( y = 6x + k \)
   c. \( y = 7x + k \)
   d. \( y = 100x + k \)
   e. \( y = qx + k \)
   f. What is the relationship between \( k \) and the slope?
Additional Practice

Lesson 4.12

1. Use elimination to solve each system of equations. Check that your solution satisfies both equations.
   a. \( x + y = 3 \)  
      \( x - y = 7 \)
   b. \( 5a - 2b = -7 \)  
      \( -5a + 6b = -9 \)
   c. \( 5x + 2y = 51 \)  
      \( 7x - 2y = 9 \)
   d. \( -9x + 3y = -2 \)  
      \( 9x - 12y = -10 \)
   e. \( 3a - 5b = 4 \)
   f. \( y = 3x - 44 \)

2. At a wash-and-fold, 2 washing machine loads and 4 dryer loads cost $11.00. Four washing machine loads and 4 dryer loads cost $16.00.
   a. What is the cost of one washing machine load?
   b. What is the cost of one dryer load?

3. a. What is the intersection point \( P \) of the graphs of \( 3x + 4y = 18 \) and \( 7x - 3y = -32 \)?
   b. Write the equation of the vertical line that contains point \( P \).
   c. Write the equation of the horizontal line that contains point \( P \).
   d. Write the equation that results when you subtract the second equation in part (a) from the first equation in part (a). Show that point \( P \) satisfies the resulting equation.

4. Write the equations of two lines with slopes \(-3\) and \(\frac{3}{4}\) that intersect at the point \((-5, 8)\).

5. Write the equation of a line that does not intersect with the line \(3x - 5y = 25\).

6. At a bagel shop, 2 coffees and 4 bagels cost $6.00. Two coffees and 7 bagels cost $8.25.
   a. What is the price of 3 bagels?
   b. What is the price of 1 bagel?
   c. What is the price of 2 coffees?
   d. What is the price of 1 coffee?
   e. What combination of coffees and bagels, if any, would cost $9.00?

7. What is the point of intersection of each pair of graphs? Check that this point satisfies both equations.
   a. \( y = 4x - 7 \)  
      \( 3x - 5y = -33 \)
   b. \( b = -5a + 8 \)  
      \( b = 11a + 104 \)
   c. \( 5p - 7q = -1 \)  
      \( -3p + 7q = 23 \)
   d. \( 3m - 5n = 36 \)  
      \( -4m - 5n = 22 \)
   e. \( 18x - 2y = 8 \)  
      \( -3x + 5y = 22 \)
   f. \( 7v - 6w = 60 \)  
      \( -6v + 8w = -60 \)

8. Solve each system.
   a. \( 5x + 3y = 8 \)  
      \( 4x - 2y = -20 \)
   b. \( 10x + 3y = 7 \)  
      \( 2x - 9y = -5 \)
   c. \( 3x - 4y = 10 \)  
      \( 9x - 12y = -10 \)
Lesson Quiz  Lesson 4.10

1. At a baseball game, Carlton buys 2 hot dogs, 1 box of popcorn, and 1 soda. His total is $8.25. Manny buys 1 hot dog, 1 box of popcorn, and 2 sodas. His total is $7.25. If a soda costs $1.50, how much does a hot dog cost?

2. At an airport, Parking Company A charges $2.00 to enter the parking lot and $8.00 per day. Parking Company B charges $10.00 to enter the parking lot and $6.00 per day.
   a. Which company is less expensive if you need to park for 3 days?
   b. Which company is less expensive if you need to park for 8 days?
   c. On what day do the parking companies charge the same amount?

3. What is the solution to each system of equations? If there is no solution, explain.
   a. $3x + y = 20$
   b. $(y + 2) = \frac{1}{2}(x - 8)$
   c. $y - 3 = 4(x + 2)$
   
   $x = 4 + y$
   $(y - 1) = \frac{1}{2}(x + 4)$
   $y - 3 = 8(x + 2)$

Lesson Quiz  Lesson 4.11

1. Identify the graphs of each pair of equations as parallel, intersecting, or identical. Explain.
   a. $4x + y = 5$ and $x - 2y = 8$
   b. $y = \frac{2}{5}x - 4$ and $y = -\frac{3}{2}x - 4$
   c. $y = \frac{2}{5}x + 3$ and $y = \frac{4}{10}x + 3$
   d. $2x - y = 3$ and $6x - 3y = 2$
   e. $y - 3 = 4x + 7$ and $y - 2x = 2x + 6$
   f. $y = 7$ and $x = 4$

2. Write an equation of the line that contains the given point and is parallel to the given line.
   a. $(2, -7); 3x + 2y = 2$
   b. $\left(10, -\frac{13}{3}\right); y - 6 = \frac{13}{3}(x - 4)$

3. Multiple Choice  Tara baby-sits and works at an ice cream store.
   • Tara baby-sits for 6 h and works at the store for 4 h. She makes $93.
   • Tara baby-sits for 3 h and works at the store for 5 h. She makes $71.25.

Which system of equations can be used to find $x$, Tara's hourly baby-sitting wage, and $y$, Tara's hourly wage working at the ice cream store?

A. $x + y = 93$
   B. $6x + 3y = 93$
   C. $6x + 4y = 93$
   D. $6x + 4y = 71.25$

$x + y = 71.25$
$4x + 5y = 71.25$
$3x + 5y = 71.25$
$3x + 5y = 93$

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Lesson Quiz

Lesson 4.12

1. Dean buys 4 candy bars and a pack of gum for $2.75. Robin buys 2 candy bars and 3 packs of gum for $2.25. What is the cost of each item?

2. Solve each system. Use elimination.
   - a. \(7x + y = 10\)
     \(2x + 3y = -8\)
   - b. \(5x + 4y = 5\)
     \(7x - 6y = 36\)
   - c. \(y = -3x + 1\)
     \(y = -5x + 2\)
   - d. \(7x + 4y = 20\)
     \(5x + 6y = 19\)

3. a. Write the equations of two lines that do not intersect.
    b. Write the equations of two lines that have every point in common.
    c. Write the equations of two lines that intersect at \((2, -1)\).

Lesson Quiz

Lesson 4.14

1. Solve each inequality. Graph your solution on a number line.
   - a. \(3x + 7 > 34\)
   - b. \(4x - 3 \leq 9x + 27\)
   - c. \(3(x - 2) > 9\)
   - d. \(3(x + 4) \geq 7x + 7\)

2. What is the solution of the inequality \(x^2 - 2x - 8 \geq 0\)?
   Use the graph shown.

3. The equation \(x^3 + 3x^2 = x + 3\) has three solutions:
   \(x = -3\), \(x = -1\), and \(x = 1\). The solutions separate the number line into four parts. Test a value in each part. Draw the solution to the inequality \(x^3 + 3x^2 > x + 3\) on the number line.
**Chapter Test**  
**Chapter 4**  
**Form A**

**Multiple Choice**

1. Consider these lines.
   \[ \ell: y = 3x \]
   \[ m: 2x - 3y = -15 \]
   What is true about point (1, 3)?
   A. It is on both lines.
   B. It is not on either line.
   C. It is on line \( \ell \) only.
   D. It is on line \( m \) only.

2. The intersection of the two lines \( y = -3x + 1 \) and \( y = 6x - 3 \) is in which quadrant?
   A. I  
   B. II  
   C. III  
   D. IV

3. What is the solution to this system of equations?
   \[ 2x + y = 3 \]
   \[ 4x - 2y = 12 \]
   A. \( \left( \frac{9}{4}, \frac{3}{2} \right) \)  
   B. \( \left( \frac{9}{4}, -\frac{3}{4} \right) \)  
   C. \( \left( -\frac{9}{4}, -3 \right) \)  
   D. no solution

4. Which pair of lines does NOT intersect?
   A. \( y = 2x - 5 \) and \( y = -2x + 5 \)
   B. \( (y + 3) = 12(x - 2) \) and \( (y - 4) = \frac{1}{12}x \)
   C. \( x - 3y = 12 \) and \( y = 4 \)
   D. \( (y + 3) = 4(x + 3) \) and \( 2y = 8x \)

5. Suppose \( x \) is greater than \(-4\). Which of these must be true?
   A. \( -x \geq -4 \)  
   B. \( 2x < 8 \)  
   C. \( -x < 4 \)  
   D. \( |x| < -4 \)

**Open Response**

6. Solve this system of equations. Show your work.
   \[ x + y = 1 \]
   \[ 4x - 2y = 10 \]
Chapter Test (continued)  

Chapter 4

Form A

7. Here is the graph of \( y = -2x^2 + 3 \).

\[
\begin{align*}
\text{a.} & \text{ Graph } y = -5 \text{ on the same coordinate plane.} \\
\text{b.} & \text{ Solve the inequality } -2x^2 + 3 \geq -5 \text{ using your graph.}
\end{align*}
\]

8. At a grocery store, bananas cost \$.90 each and oranges cost \$.80 each. Ava has \$15 to spend.
   \a. If Ava wants to buy exactly 5 bananas, at most how many oranges can she buy?
   \b. If Ava wants to buy an equal amount of bananas and oranges, at most how many of each can she buy?

9. At a store, ten bottles of soda and five bags of chips cost \$22.50. Two bottles of soda and four bags of chips cost \$11.10. What is the cost of a single bottle of soda? Show your work.

10. Graph the two lines. Find the exact coordinates of their point of intersection.
    
    \[
    \begin{align*}
    x - 3y &= \frac{21}{2} \\
y &= -2x
    \end{align*}
    \]

11. Graph the solution set to the inequality \( 2x + 14 \geq -3 \).

Challenge Problem

12. Draw a graph of \( y = |2x + 4| \).
   \a. Does the point \( O(-2, 0) \) lie on the graph? Explain.
   \b. Suppose \( A(a, -2a - 4) \) where \( a < -2 \) and \( B(b, 2b + 4) \) where \( b > -2 \). Show that \( m(A, O) = -m(O, B) \).
Multiple Choice

1. Consider these lines.
   \[ m: 3x - 4y = 10 \]
   \[ n: y = -\frac{x}{2} \]
   What is true about point (2, -1)?
   A. It is on both lines.
   B. It is not on either line.
   C. It is on line \( m \) only.
   D. It is on line \( n \) only.

2. The intersection of the two lines \( y = -4x - 6 \) and \( y = x + 9 \) is in which quadrant?
   A. I   B. II   C. III   D. IV

3. What is the solution to this system of equations?
   \[ x - 3y = -4 \]
   \[ 2x + y = -1 \]
   A. (-1, -1)   B. (1, 1)   C. (-1, 1)   D. no solution

4. Which pair of lines does NOT intersect?
   A. \( y + 5x = -7 \) and \( y = 5x + 2 \)
   B. \( 3x + 7y = -2 \) and \( 7x + 3y = 2 \)
   C. \( (y - 1) = -4(x - 8) \) and \( y = \frac{x}{4} \)
   D. \( 10x - 2y = 0 \) and \( y - 3 = 5(x + 1) \)

5. Suppose \( x \) is less than or equal to 6. Which of these must be true?
   A. \( -x \geq -6 \)   B. \( 6 < x \)
   C. \( 6 \leq x \)   D. \( x \geq -6 \)

Open Response

6. Solve this system of equations. Show your work.
   \[ x - y = 8 \]
   \[ 3x + 2y = -21 \]
Chapter Test (continued)  Chapter 4
Form B

7. Here is the graph of \( y = x^2 - 2 \).

![Graph of \( y = x^2 - 2 \)]

a. Graph \( y = 7 \) on the same coordinate plane.
b. Solve the inequality \( x^2 - 2 \leq 7 \) using your graph.

8. At a garden store, rose bushes cost $12 each and holly bushes cost $8 each. Aiden has $60 to spend.
   a. If Aiden wants to buy exactly 4 rose bushes, at most how many holly bushes can he buy?
   b. If Aiden wants to buy an equal amount of rose bushes and holly bushes, at most how many of each can he buy?

9. At a store, 24 bottles of water and 6 bags of chips cost $23.40. Five bottles of water and two bags of chips cost $6.60. What is the cost of a single bottle of water? Show your work.

10. Graph the two lines. Find the exact coordinates of their point of intersection.
    \[
    2x - y = -2 \\
    y = x - 1
    \]

11. Graph the solution set to the inequality \( 4 - 3x > -17 \).

Challenge Problem

12. Draw a graph of \( y = |5x - 10| \).
   a. Does the point \( O(2, 0) \) lie on the graph? Explain.
   b. Suppose \( A(a, -5a + 10) \) where \( a < 2 \) and \( B(b, 5b - 10) \) where \( b > 2 \). Show that \( m(O, A) = -m(O, B) \).
Implementing and Teaching Guide
The Center for Mathematics Education Project was developed at Education Development Center, Inc. (EDC) within the Center for Mathematics Education (CME), with partial support from the National Science Foundation.

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A Letter From Al Cuoco

Dear Colleagues,

We have had the CME Project under development for five years. However, the genesis of its philosophy and approach goes back much further than that. In a real sense, I’ve been working on the ideas in this program for close to four decades.

Like many teachers, I was always dissatisfied with the texts I used. I wanted books that conveyed the true spirit of doing mathematics—books that didn’t get bogged down in vocabulary and convention, and books that put school mathematics into the broader landscape of mathematics as a scientific discipline.

The CME Project has given my colleagues and me a chance to build such a program. We have structured it around a few core principles:

• **Habits of Mind**
  For many high school students, the real utility of mathematics lies in a style of work—the habits of mind that allow one to look at the world through a mathematical lens. The program features these mathematical habits throughout.

• **Experience Before Formality**
  Careful definitions and well-worded theorems are important, but students need to grapple with ideas and problems—I like to say “muck around in the mathematics”—before they bring ideas to closure.

• **High Expectations**
  The vast majority of students have the capacity to think in ways that are characteristically mathematical.

• **Textured Emphasis**
  In the CME Project, we are careful to separate convention and vocabulary from matters of mathematical substance. We have even designed our practice problems so that they have a larger mathematical point.

• **General-Purpose Tools**
  Fluency in algebraic calculation, proof, and graphing are essential in mathematics, but special-purpose techniques like FOIL and the rote application of \( y = mx + b \) are not.

• **A Mathematical Community**
  Our writers, field testers, reviewers, and advisors come from all parts of the mathematics community—teachers, mathematicians, education researchers, technology developers, and administrators. Our team hopes that you find the following pages useful and stimulating. And we hope that you have as much fun working through the mathematical development as we had creating it.

—Al Cuoco
For the CME Project team
Introduction to the CME Project

The CME Project, developed by the Center for Mathematics Education (CME) of the Education Development Center (EDC), is a new NSF-funded high school program. It is organized around the familiar courses of algebra 1, geometry, algebra 2, and precalculus. The CME Project provides teachers and schools with a third alternative to the choice between traditional texts driven by basic skill development and more progressive texts that have unfamiliar organizations.

This program gives teachers the option of a problem-based, student-centered program, organized around the mathematical themes with which teachers and parents are familiar. Furthermore, the tremendous success of NSF-funded middle school programs has left a need for a high school program with similar rigor and pedagogy. The CME Project fills this need.

The goal of the CME Project is to help students acquire a deep understanding of mathematics. Therefore, the mathematics here is rigorous.

We took great care to create lesson plans that, while challenging, will capture and engage students of all abilities and improve their mathematical achievement.

About CME

EDC’s Center for Mathematics Education, led by mathematician and teacher Al Cuoco, brings together an eclectic staff of mathematicians, teachers, cognitive scientists, education researchers, curriculum developers, specialists in educational technology, and teacher educators, internationally known for leadership across the entire range of K–16 mathematics education. We aim to help students and teachers in this country experience the thrill of solving problems and building theories, understand the history of ideas behind the evolution of mathematical disciplines, and appreciate the standards of rigor that are central to mathematical culture.

CME Project Core Team

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2. Al Cuoco
3. Kevin Waterman
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5. Nancy Antonellis D’Amato
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Absent from photo: Daniel Erman, Brian Harvey, Stephen Maurer, Audrey Ting

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Contributors to the CME Project

National Advisory Board
The National Advisory Board met early in the project, providing critical feedback on the instructional design and the overall organization. Members include
Richard Askey, University of Wisconsin
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Core Mathematical Consultants
Dick Askey, Ed Barbeau, and Roger Howe have been involved in an even more substantial way, reviewing chapters and providing detailed and critical advice on every aspect of the program, from the logical organization to the actual numbers used in problems. We can’t thank them enough.

Teacher Advisory Board
The Teacher Advisory Board for the CME Project was essential in helping us create an effective format for our lessons that embodies the philosophy and goals of the program. Their debates about pedagogical issues and how to develop mathematical topics helped to shape the distinguishing features of the curriculum so that our lessons work effectively in the classroom. The advisory board includes
Jayne Abbas, Richard Coffey, Charles Garabedian, Dennis Geller, Eileen Herlihy, Doreen Kilday, Gayle Masse, Hugh McLaughlin, Nancy McLaughlin, Allen Olsen, Kimberly Osborne, Brian Shoemaker, and Benjamin Sinwell

Field-Test Teachers
Our field-test teachers gave us the benefit of their classroom experience by teaching from our draft lessons and giving us extensive, critical feedback that shaped the drafts into realistic, teachable lessons. They shared their concerns, questions, challenges, and successes and kept us focused on the real world. Working with these expert professionals has been one of the most gratifying parts of the development—they are “highly qualified” in the most profound sense.

California Barney Martinez, Jefferson HS, Daly City; Calvin Baylon and Jaime Lao, Bell JHS, San Diego; Colorado Rocky Cundiff, Ignacio HS, Ignacio; Illinois Jeremy Kahan, Tammy Nguyen, and Stephanie Pederson, Ida Crown Jewish Academy, Chicago; Massachusetts Carol Martignette, Chris Martino, and Kent Werst, Arlington HS, Arlington; Larry Davidson, Boston University Academy, Boston; Joe Bishop and Carol Rosen, Lawrence HS, Lawrence; Maureen Mulryan, Lowell HS, Lowell; Felisa Honeyman, Newton South HS, Newton Centre; Jim Barnes and Carol Haney, Revere HS, Revere; New Hampshire Jayne Abbas and Terin Voisine, Cawley Middle School, Hooksett; New Mexico Mary Andrews, Las Cruces HS, Las Cruces; Ohio James Stallworth, Hughes Center, Cincinnati; Texas Arnell Crayton, Bellaire HS, Bellaire; Utah Troy Jones, Waterford School, Sandy; Washington Dale Erz, Kathy Greer, Karena Hanscom, and John Henry, Port Angeles HS, Port Angeles; Wisconsin Annette Roskam, Rice Lake HS, Rice Lake
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Hi, I’m Annette Roskam. I field-tested the CME Project Algebra 1 materials over two years in Rice Lake, Wisconsin. Like many of us, I have a “procedural” teaching background:

Here’s the slope formula. Use it.

Here are three forms of an equation of a line. Plug in the numbers you know and write the equation.

So, I was still uncomfortable with this totally new CME-Project way of engaging students as we headed into Algebra 1 Lesson 4.6 (Equations of Lines).

On that day, I planned for my classes to work through Lesson 4.6 to learn how to write equations of lines. In one class (not my top class, either), the students asked how we might write the point-tester for an equation. They ended up by asking about simplifying the point-tester so it became a “nice-looking” equation, meaning one with a simple form such as

\[ y = \frac{3}{5}x + 6. \]

Knowing something about this form that they didn’t, I asked what they noticed. They said they saw the slope and y-intercept.

I wondered aloud what the graph might look like and whether we needed a table of values to find out. Someone suggested plotting (0, 6) and using slope.

My goodness! Wanting to keep this opportunity alive, I suggested we look at another equation. We did. They noticed the same thing!

I gave them five more equations in a variety of forms and asked them to identify slope and y-intercept. They got every single one of them right!

I suggested we go back to the lesson to write more equations and check out what Tony and Sasha were considering. We checked to see whether using either point would give the same equation. Voilà, they did!

We were steamrolling. That lesson was in the bag. The next day was within reach. I asked what the graph would look like. They told me.

I wanted to water this flower. We took Tony and Sasha’s equation.

- We wrote steps to change from point-tester form to simplified \( y = \) form.
- We determined what we could see in each equation.
- We talked about how we might graph an equation in that form.

They didn't miss a step!

It was time to wrap up the lesson. I asked how we might name this form of the equation. They said—as must have some mathematician long ago and far away—“point-slope” and “slope-intercept.”

—Annette Roskam
Program Overview

In a Nutshell
The organization of the Center for Mathematics Education (CME) Project provides students the time and focus they need to develop fundamental mathematical ways of thinking. Its primary goal is to develop in students robust mathematical proficiency.

- The program employs innovative instructional methods, developed over decades of classroom experience and informed by research, that help students master mathematical topics.
- One of the core tenets of the CME Project is to focus on developing students’ Habits of Mind, or ways in which students approach and solve mathematical challenges.
- The program builds on lessons learned from high-performing countries: develop an idea thoroughly and then revisit it only to deepen it; organize ideas in a way that is faithful to how they are organized in mathematics; and reduce clutter and extraneous topics.
- It also employs the best American models that call for grappling with ideas and problems as preparation for instruction, moving from concrete problems to abstractions and general theories, and situating mathematics in engaging contexts.
- The CME Project is a comprehensive curriculum that meets the dual goals of mathematical rigor and accessibility for a broad range of students.

Essentially, we have come to the position that the real utility of mathematics—both for students who specialize in mathematically related fields and for those who take other directions—lies in a style of work, a collection of mathematical habits of mind, which mathematicians use to make sense of the world. Such mathematical habits include

- visualization
- performing thought experiments
- reasoning by continuity or linearity
- mixing deduction with experiment

The basic results and methods of high school mathematics—the Pythagorean Theorem, solving equations, graphing lines, analyzing data, and so on—are the products of mathematics. The actual mathematics lies in the thinking that is used to discover and develop these results. It is essential to develop both the results and the thinking.

Our entire program, from our uses of technology to the design of the exercises for each lesson, is devoted to students becoming mathematical thinkers as they develop mathematical understanding.

While an emphasis on mathematical thinking was at the core of our motivation when we applied to the NSF to develop the CME Project, there have been other motivations as well.

- The field demands an innovative program with a familiar structure.

Studies have shown that many teachers, parents, and school districts want a problem-based, student-centered program that maintains the familiar American course structure of algebra 1, geometry, algebra 2, and precalculus.

- That structure allows us to focus on habits of mind.

Habits of mind, like all habits, take time to develop. Our algebra courses, although they

Basic Principles
The actual development of the CME Project began in 1992 with a grant from the National Science Foundation (NSF) to develop a standards-based geometry course. But the basic principles on which the program rests have evolved over nearly four decades.
consider topics from many fields, focus on algebraic thinking. We revisit some of the topics in geometry and in precalculus, where we look at them from a more analytic perspective.

- **We want a program that has involvement of the whole mathematical community.**

Our advisors, core consultants, and writers include research mathematicians with longstanding interests in precollege mathematics. Core consultants have worked with us to outline chapters, design lessons, create exercises, and develop approaches to topics.

- **We have decades of experience with classroom-effective methods.**

We work closely with teachers—as advisors, field testers, and writers—as well as with mathematicians. The core CME Project writing staff has more than a century of combined K–16 teaching experience. We have developed particularly effective approaches to topics that are notoriously difficult for students—algebra word problems, graphing, function algebra, complex numbers, proof, and data analysis, among many others.

- **We want a program that helps students bring mathematics into their world.**

“Power users” of mathematics are able to see abstract connections among seemingly different phenomena and can synthesize mathematical methods, often in unorthodox ways. The first step in developing this proficiency is to expand students’ conceptions of the real world to include mathematics.

Students in the CME Project apply elementary algebra and mathematical induction to determine the monthly payment on a loan. They use complex numbers as a device for establishing trigonometric identities. They apply Euclidean geometry to perspective drawing, optimization problems, and trigonometry.

All these situations are applications of mathematics, because the emphasis is on how one uses mathematics, as opposed to where one uses it.

- **We want our program to be free of all the special-purpose tools that clutter the curriculum.**

Popular texts, curricula, and the traditions of high school teaching are full of ad hoc methods and techniques that may work locally but that have little or no application after the chapter at hand is finished.

Here are a few that veteran teachers will recognize.

- The “box method” for setting up word problems—a method that works only for problems designed to yield to it!
- Methods for graphing (or finding equations) that work only for lines
- Expanding methods such as FOIL—a method that works only for multiplying binomials
- Factoring methods that work only on quadratics

These and many other methods in mathematics have no counterparts outside secondary school. And they are extremely inefficient. Students need to unlearn them when they need to solve problems (on high-stakes exams, for example) that do not fit the constraints of the methods.

The CME Project develops general-purpose tools and approaches that teachers can introduce in elementary algebra and extend as students gain mathematical sophistication. These tools are based on an analysis of the habits of mind used by mathematicians, on the classroom experiences of many middle school and high school teachers, and on a sound epistemological theory of how mathematical ideas develop.

- **We want a program with high expectations for students.**

We have seen that students at all levels can do serious mathematics. Much of the field-testing of the CME Project Algebra 1 took place in ordinary and low-performing
classes. In these, and indeed all classes, the vast majority of students are able to think in characteristically mathematical ways.

Lack of access to appropriate mathematics, not a lack of ability to think mathematically, is a prime cause for the poor performance in mathematics one so often sees. The CME Project design employs a low-threshold, high-ceiling approach. Each chapter starts with activities that are accessible to all students. Each chapter ends with exercises that will challenge the most advanced students.

It is pleasantly surprising to see how far students often take the materials.

**Instructional Design**

The design of each course in the CME Project reflects our commitment to a tight, coherent set of vertically aligned materials:

**Chapters**

Each book contains eight chapters. Every chapter has easy entry points and is organized around a large mathematical theme.

**Investigations**

Each chapter contains a number of Investigations that develop the chapter theme extensively.

**Lessons**

Each Investigation consists of several Lessons. The exploratory first lesson, Getting Started, includes exercises that are to be tried—but not necessarily completed—before instruction. These exercises preview the main ideas of the Investigation, provide experiments that students can use in the Investigation, activate prior knowledge that students will need for the Investigation, and allow the teacher to check what students bring to the Investigation.

The remaining lessons include exposition, instruction, and exercises on the core topics.

The instruction is organized around the following elements.

- **In-Class Experiment** provides a series of questions, procedures, and calculations for students to work on, typically in groups. It provides an experience that should help students understand the basis of a key concept or formula.

- **Developing Habits of Mind** provides multiple ways to think about some lesson content, often emphasizing a habit that has applications beyond the mathematics at hand.

- **Minds in Action** is a conversation among fictional students that relates their attempts (successful or not) to solve a problem. These dialogs model students practicing good habits of mind.

- **For Discussion** typically follows a Minds in Action or some other important exposition. It presents questions or problems for the whole class to discuss. Its purpose is to make sure students have read and understood the text.

- **Example** is a worked-out problem that illustrates an instructional point or helps students codify what they already know.

- **For You To Do** typically follows an Example, a Minds-in-Action dialog, or some other important exposition. It provides, for students to try in class, a problem that is related to a central idea just discussed.

- **Definition** gives a formal mathematical definition, typically after some form of experimentation.

- **Theorem** and **Corollary** give a formal mathematical fact, typically, as with a Definition, after students have had experience with specific numerical examples. Proofs accompany Theorems and Corollaries, unless students are asked to provide their own proof.
Facts and Notation describe symbols, sets, or other concepts, where a formal definition would not be appropriate.

Conjecture is a plausible statement that students are not yet ready to prove, but which they will prove in the current course.

Assumption is a plausible statement that cannot be proved or will not be proved in the current course. Students are to accept it as true, nonetheless.

Side Notes beside the exposition remind students to experiment, to visualize, or to try to represent a rule; in general to use good habits of mind.

The exercises, while building students’ computational skills, are also designed to reinforce strong mathematical habits of mind. They occur in three groups.

For You to Explore problems and Check Your Understanding exercises are to be done in class.

On Your Own exercises provide homework.

Maintain Your Skills exercises provide students with practice on important skills and can also lead to new conjectures.

The exercises feature certain types.

Write About It exercises require students to write about their strategies or about where they get stuck.

What’s Wrong Here? exercises ask students to find, describe, and correct mistakes—mistakes that typically highlight common errors.

Take it Further exercises are optional and extend or enrich topics beyond the core of the course.

Standardized Test Prep exercises mimic types of questions students can expect to find on high-stakes tests, such as state exit exams.

Some lessons end with a Historical Note about the history behind the mathematics being studied.

Each chapter ends with a Project. Teachers can assign the Project early in the chapter and allow students to take as long as the chapter takes to complete it.

The teacher support materials give advice about various jumping-off points for a chapter, so that teachers can customize the chapter to individual classes.
As suggested in the Program Overview, the following organizing principle is fundamental to development of the CME Project.

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more important, from developing the ways of thinking—the *habits of mind*—used to create the results.

To be more concrete about what we mean by *habits of mind*, we adapted the following piece from “Habits of Mind.” For a complete treatment, see the full paper.

A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics say and what they do.

Such a curriculum lets students in on the process of creating, inventing, conjecturing, and experimenting. It lets them experience what goes on behind the study door before new results are polished and presented. It is a curriculum that encourages false starts, calculations, experiments, and explaining special cases. Students develop the habit of reducing things to lemmas for which they have no proofs, and of suspending work on these lemmas and on other details until they see if assuming the lemmas are true will help. It helps students look for logical and heuristic connections between new ideas and old ones.

A habits-of-mind curriculum is devoted to giving students a genuine research experience.

In high school, the CME Project would like students to acquire

- some useful general habits of mind
- some mathematical approaches that have shown themselves worthwhile over the years

These are general approaches. Also, there are content-specific habits that high school graduates should have. The CME Project concentrates on two of the several possible categories:

- some geometric habits of mind that support the mathematical approaches
- some algebraic ways of thinking that complement the geometric approaches

Behind the CME Project is the belief that every course or academic experience in high school should be used as an opportunity to help students develop good general habits of mind.

These general habits of mind are not the sole province of mathematics. The research historian, the house builder, and the mechanic who correctly diagnoses what ails your car all use them. Nor are they guaranteed byproducts of learning mathematics. The major lament of reform efforts is that it has been shown possible for students to learn the facts and techniques developed by mathematicians (historians, auto diagnosticians, . . . ) without ever understanding how mathematicians (or these others) think.

Good thinking must apparently be relearned in a variety of domains, but further remarks here will be specific to the domain of mathematics.

So, a key goal of the CME Project curriculum is for students to think about mathematics the way mathematicians do. Experience shows that they can. This does not mean that high school students should be able to
understand the topics that mathematicians investigate, of course. It does mean that high school graduates should be accustomed to using real mathematical methods.

They should be able to use the research techniques that have been so productive in modern mathematics.

They should be able to develop conjectures and provide supporting evidence for them.

When asked to describe mathematics, they should say that it is about ways of solving problems. They should no longer say that it is about triangles, solving equations, or finding percents.

The danger of wishing for this is that it is all too easy to turn ways of solving problems into a curriculum that drills students in The Five Steps for Solving a Problem. That is not what the CME Project is after. The CME Project is after mental habits that allow students to develop a repertoire of general heuristics and approaches that they can apply in many different situations.

In the following sections, you will see the word should very often. Take it with a grain of salt. The CME Project is intended to develop a repertoire of useful habits. The most important of these is the understanding of when to use what.

Students Should be . . . Pattern Sniffers

Criminal detection, the analysis of literature or historical events, and the understanding of personal or national psychology all require you to be on the lookout for patterns.

In the context of mathematics, students should learn to delight in finding hidden patterns—for example, in a table of the squares of the integers between 1 and 100. Students should be always on the lookout for shortcuts that arise from patterns in calculations (summing arithmetic series, for example).

Students should fall into the habit of looking for patterns when they are given problems by someone else. (“Which primes are the sums of two squares?”) A search for regularity should drive the kinds of problems students pose for themselves, convincing themselves, for example, that there must be a test for divisibility by 7. Pattern sensitivity should extend to their daily lives.

Students can develop the habit of comparing structures (finding isomorphisms, in fact). For example, the operation of taking the union of two sets looks very much like the operation of taking the sum of two numbers.

. . . Experimenters

Performing experiments is central in mathematical research, but experimenting is all too rare in mathematics classrooms.

Simple ideas such as recording results, keeping all but one variable fixed, trying very small or very large numbers, and varying parameters in regular ways are missing from the backgrounds of many high school students. Non-Euclidean geometry got its start when mathematicians took an existing system and changed one feature.

When faced with a mathematical problem, a student should immediately start playing with it, using strategies that have proved successful in the past. Students should be used to performing thought experiments, so that, without writing anything down, they can give evidence for their answers to questions such as “What kind of number do you get if you square an odd number?”

Students should also develop a healthy skepticism about experimental results. Results from empirical research can often suggest conjectures, and occasionally they can point to theoretical justifications. But mathematics is more than data-driven discovery, and students need to realize the limitations of the experimental method.
. . . Describers
Many people claim that mathematics is a language. But it is more than a spoken language in that it contains extra constructs and symbols. It allows you to invent, on the fly, new and expressive words and descriptions.

Students should develop some expertise in playing the mathematics language game. They should be able to do things like

Give precise descriptions of the steps in a process. Describing what you do is an important step. A great deal of what is called mathematical sophistication comes from the ability to say exactly what you mean.

Invent notation. One way for students to see the utility and elegance of traditional mathematical formalism is for them to struggle with the problem of describing phenomena for which ordinary language descriptions are much too cumbersome (combinatorial enumerations, for example).

Argue. Students should be able to convince classmates that a particular result is true or plausible by giving precise descriptions of good evidence or (even better) by showing generic calculations that actually constitute proof.

Write. Students should develop the habit of writing down their thoughts, results, conjectures, arguments, proofs, questions, and opinions about the mathematics they do. They should be accustomed to polishing up their notes every now and then for presentation to others.

Formulating oral and written descriptions of your work is useful when you are part of a group of people with whom you can trade ideas. Part of students’ experience should be a culture in which they work in collaboration with each other and feel free to ask questions of each other and comment on each other’s work.

. . . Tinkerers
Tinkering is at the heart of mathematical research.

Students should develop the habit of taking ideas apart and putting them back together. When they do this, they should want to see what happens if something is left out or if the pieces are put back in a different way.

After experimenting with a rotation followed by a translation, they should wonder what happens if a translation is followed by a rotation.

When they see that every integer is the product of primes, they should wonder, for example, if every integer is the sum of primes.

Rather than walking away from the “mistake”

\[
\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}
\]

they should ask,

- Are there any fractions for which this statement is true?
- Are there any sensible definitions for a binary operation for which this statement would be true?

. . . Inventors
Tinkering with existing machines leads to expertise at building new ones.

Students should develop the habit of inventing mathematics both for utilitarian purposes and for fun. Their inventions might be rules for a game, algorithms for doing things, explanations of how things work, or even axioms for a mathematical structure.

Good mathematical inventions, like most good inventions, give the impression of being innovative but not arbitrary. Even rules for a game, if the game is to intrigue anyone, must have an internal consistency and must make sense.

For example, a baseball rule that required runners arriving at second base to stop
and jump up and down five times before continuing to third base would be useless because it would not fit with the rest of the game. No one would stand for it. It is not a very interesting invention.

The same could be said of those “math team” problems that ask you to investigate the properties of some silly binary operation that seems to fall out of the sky, such as $\diamond$, where

$$a \diamond b = \frac{a + 2b}{3}$$

It is a common misconception that mathematicians spend their time writing down arbitrary axioms and deriving consequences from them. Mathematicians do enjoy deriving consequences from axiom systems they invent. But in fact the operation $\diamond$, and other axiom systems, always emerge from the experiences of the inventors.

They arise in an attempt to bring some clarity to a situation or to a collection of situations. For example, consider the following game. (For some, it is more than a game.)

Person A offers to sell person B something for $100. Person B offers $50. Person A comes down to $75. Person B offers $62.50. They continue haggling in this way, each time taking the average of the previous two amounts. On what amount will they converge?

This is a concrete problem. Its solution leads to a general theorem:

If person A starts the game at $a$ and person B makes an offer of $b$, the haggling will lead to the limit

$$a + 2b$$

$$\frac{3}{3}$$

This might lead you to define the binary operation $\diamond$ by

$$a \diamond b = \frac{a + 2b}{3}$$

and to derive some of its properties. (For example, $a \diamond b$ is closer to $b$ than it is to $a$. This property explains why a salesperson never tells you how much a car costs until you make a first offer.) Now, the invention of $\diamond$ no longer seems arbitrary, even though the consequences of the definition might become quite far removed from the situation that motivated it.

The practice of inventing a mathematical system that models a particular phenomenon is crucial to the development of mathematics. One reason for this is that the mathematical models often find utility outside the situations that produced them.

A vector space is a classic example. The notion originated to describe directed line segments in two and three dimensions. It turned out that many other mathematical objects (polynomials, matrices, and complex numbers, for example) form vector spaces.

. . . Visualizers

There are many types of visualization habits in mathematics. Here are three:

First, you may visualize things that are inherently visual—doing things with mental images or actual diagrams. For example, you might approach the question “How many windows are there in your house or apartment?” by constructing a mental picture and manipulating the picture in various ways.

Second, you may construct visual analogs to ideas or processes that you first encounter in nonvisual realms. This includes, for example, using an area model to visualize multiplication of two binomials (or any two numbers, such as 23 and 42 or $\frac{31}{2}$ and $\frac{11}{3}$).

The purpose of such an analog may be to aid understanding of the process, or merely to help you keep track of a computation.

Third, you may employ visual accompaniments (not analogs, exactly) to totally nonvisual processes. To multiply binomials for example, you might actually picture the symbols moving about in some orderly fashion to help structure the computation. The imagery may not
clarify meaning—it may just support the task, focus your attention, or the like—but such visualizations do become part of a mathematician’s repertoire.

You can subdivide these three visualization habits a bit more finely. You get categories like these.

**Visualize data.** Students should construct tables and graphs. They should use these visualizations in their experiments.

**Draw diagrams.** Students should customarily use sketches or three-dimensional imagery to work with ideas in which size is irrelevant (Venn diagrams and factor trees, for example) or arbitrary (the stuff of classical geometry, extended to include three dimensions), or things are too small, too large, or too diverse to be seen.

**Visualize relationships.** Students should think in terms of machines. All kinds of visual metaphors (food processors, coin-operated machines, function machines, specialized calculators, and so on) support this kind of imagery. Students should also use many visual representations for the input-output pairing associated with a function, including ordinary Cartesian graphs for functions from real numbers to real numbers.

**Visualize change.** Seeing how a phenomenon varies continuously is one of the most useful habits of classical mathematics. Sometimes the image is one that simply moves between states, as when you think of how a cylinder of fixed volume changes as you increase the radius. Other times, one thing blends into another. Think of the many demonstrations that show ellipses becoming hyperbolas. This habit cuts across many of the others, including some that seem to deal with explicitly discrete phenomena.

**Visualize calculations.** There’s a visual component to “mental arithmetic” and estimation that is often ignored. Students should be in the habit of visualizing calculations (numerical and algebraic), perhaps by seeing numbers flying around in some way. A particularly useful habit in arithmetic is to picture what an integer looks like factored into primes.

... **Conjecturers**

The habit of making plausible conjectures takes time to develop, but it’s central to the doing of mathematics.

Students should at least be in the habit of making data-driven conjectures (about patterns in numbers, for example). Ideally, however, their conjectures should rest on something more than experimental evidence. For example, in combinatorics, students experiment and then make conjectures about general counting methods.

* * * * *

In the CME Project, the development of the mathematics leads to good mathematical habits in multiple ways.

For example, a key habit in algebraic thinking is abstracting regularity from repeated calculations. To help develop this habit, students learn the guess-check-generalize method for building equations from situations.

The method involves taking several guesses, checking those guesses against the text of the problem, and carefully keeping track of the check steps you follow. Finally, students generalize with an arbitrary number, the variable, to build the equation. (They proceed from there to solve the equation and solve the problem.)

Besides learning a general-purpose tool for solving word problems—an approach that works for all kinds of word problems—they build several mathematical habits: taking guesses, keeping track of steps, and using algebra to encapsulate the work they do to check their guesses. (For more details on the guess-check-generalize approach, see pages 36–37.)
History, Research, and Development

History of the CME Project

The CME Project is a descendent of two previously-developed courses, each using traditional course structure and each focusing on mathematical thinking.

We developed *Connected Geometry* immediately after the release of the 1989 NCTM Standards.

It reflected many of the attitudes about curriculum that arose in that period of American education. The book contained few stated theorems and definitions, and even fewer worked-out examples. The theorems, definitions, and examples were all there, but they were in the teacher's edition or the solution manual. What students saw was a collection of activities and provocative problems that were designed to help them discover results for themselves.

Teachers who were not part of the original field tests told us that *Connected Geometry* was too much of a guide and not enough of a reference. They loved the open-ended problems, but they felt that the activities needed more closure in the student text.

We developed *Mathematical Methods*, a precalculus text, around the time that NCTM was revising the Standards in preparation for *Principles and Standards of School Mathematics*.

It, too, was a product of its time. It was influenced by the growing sense that all students needed technical fluency (with algebraic and numerical calculations) and that students should be able to refer to their text as a resource for results and examples.

We included many more proved theorems, worked-out examples, and practice exercises in *Mathematical Methods* than in *Connected Geometry*.

* * *

For the CME Project, we developed a structure that is informed by our previous work and faithful to both needs. Students can use their texts as both a guide and a reference.

We came to realize that, at the high school level, understanding develops as a result of two important processes.

- independent (or guided) investigation
- reading, discussing, and internalizing mathematical exposition

Each CME Project Investigation begins with a Getting Started lesson that students do before instruction. The Getting Started lesson provides experiments that preview—in simple numerical and geometric contexts—the important ideas in the exposition.

The lessons that follow then include worked-out examples or written dialogs that codify methods, bring closure to the experimentation, and provide a reference for later work.

Each lesson has a set of orchestrated practice problems. In them, students practice arithmetic and algebraic skills while they try to locate some pattern that suggests an interesting mathematical result.
Research Behind the CME Project Approach

Current work in the field of education led us to four research-based goals for the CME Project.

1. To ensure that students who complete the CME Project demonstrate a high level of mathematical proficiency
2. To provide a coherent and rigorous curriculum, based on current learning research and world-class best practices, with a focus on the central themes in mathematics, for teachers who desire the traditional sequence of high school courses
3. To provide a curriculum that will help more students succeed in four years of rigorous yet accessible mathematics in high school
4. To give students the experience of working as mathematicians and scientists; to highlight the profound utility of modern mathematical methods in mathematics, fields related to mathematics, and everyday life

Mathematical Proficiency

National reports suggest that a modern mathematics curriculum should help students develop a broad mathematical proficiency. The National Research Council (NRC) report *Adding It Up* describes this proficiency as the interweaving of five modes of thought that are characteristically mathematical.

- **Conceptual understanding**: comprehension of mathematical concepts, operations, and relations
- **Procedural fluency**: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **Strategic competence**: ability to formulate, represent, and solve mathematical problems
- **Adaptive reasoning**: capacity for logical thought, reflection, explanation, and justification
- **Productive disposition**: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy

What Mathematics?

One guiding principle, then, for choosing what mathematics to develop, and how to develop it, is how well that choice helps students build proficiency.

A second guiding principle is reflected in the definition of coherence from “A Coherent Curriculum.” The mathematics taught in school should be faithful to mathematics as a discipline.

Our emphasis on mathematical thinking and proficiency is central to staying faithful to mathematics. But the content of the program must also act in concert with mathematics—the topics, the emphases, the sequencing, and even the way we describe historical developments and the evolution of ideas.

Mathematics as a Tool

Often in high school, *applications* means using a narrow mathematical result (a theorem or a formula, say) to solve a problem outside of mathematics. In reality, such applications are rare.

Mathematical proficiency gives students the techniques and habits of mind needed to model a wide range of phenomena, in and out of mathematics. Mathematics provides "powerful tools for analytical thought as well as the concepts and language for precise quantitative description of the world around us. It affords knowledge and reasoning of extraordinary subtlety and beauty, even at the most elementary levels." In our previous curriculum work, we have seen that most high school students appreciate the “reasoning of extraordinary subtlety and beauty” afforded by mathematics when they are put in problem situations that ask them to bring together common sense, mathematical theory, and technical skill. Students (even "low-performing" students) in CME Project field...
tests reveled in the chance to develop and display their own mathematical thinking.

Learn From Top-Performing Countries
In an analysis of the curricula in top-performing countries,\textsuperscript{8} we found some commonalities that are in direct contrast to many American curricula.

In what Schmidt and his colleagues call the “A+ countries,”\textsuperscript{12} we find

- There is a small number of core themes that support the overall curriculum structure.
- Topics develop from particulars to deeper structures that are inherent in mathematics.
- Programs build prerequisite knowledge when it is needed.
- Topics do not reappear endlessly throughout a program.
- Curriculum organization mirrors the organization of the discipline.

Nonintegrated ≠ Disconnected
There are good reasons for spending an extended amount of time on classical mathematical themes. Habits of mind are just that—habits. And habits take time to develop. The ability to tease out commonalities in repeated calculations, a hallmark of algebraic thinking, requires sustained exposure and focused work.

The CME Project organization of algebra 1, geometry, algebra 2, and precalculus signifies a concentrated focus for each year, but the focus is used to study a broad spectrum of topics from mathematics and other fields.

For example, the CME Project sees geometry as the hub of a wheel, with spokes connecting to other parts of mathematics and science. The program looks at the landscape through a geometric lens and looks at geometry by using tools from algebra, arithmetic, and even physics.

When students solve the problem of positioning an airport so that the sum of the distances to three given cities is minimized, they use reasoning by continuity (and dynamic geometry software), algebra, functional thinking, classical geometry, and physics to make their conjectures.

The CME Project revisits certain core scenarios in mathematics and other fields several times over four years, each time taking a new perspective. For example, the problem of determining the monthly payment on a loan appears throughout the program. Over the course of four years, students use this context as either a launch into or an application of trial-and-error approximations, recursively defined functions, computer algebra systems, geometric series, and exponential functions.

Pedagogical Structures
The pedagogical structures imposed by the program materials serve the primary goal of increasing mathematical proficiency.

Recent research\textsuperscript{6,7} suggests a structure that

- requires students to wrestle with problems as motivation and preparation for instruction
- uses multiple contexts to support flexible knowledge transfer
- gives explicit emphasis to extracting underlying mathematical themes in these contexts to help students to develop expertise in identifying “conditions of applicability” (as defined in \textit{How People Learn}\textsuperscript{5})
- integrates solid skill building and knowledge acquisition with activities that require adaptive reasoning, abstraction, and problem solving
Development of the CME Project Approach

While developed for K–8 mathematics, the NRC taxonomy gives specificity to mathematical mindedness at all precollege levels, and our previous work convinces us that most students in high school can achieve mathematical proficiency as defined here. The CME Project used this taxonomy as a central benchmark for deciding which topics to develop and how to develop them.

CME Project Geometry, for example, develops area formulas by having students construct algorithms to dissect polygons into rectangles. As students develop proficiency at calculating area, they also develop ideas about algorithms and composition. They then use area formulas to explain and prove other results, from triangle similarity tests to the fact that the sum of the distances to the sides of an equilateral triangle from a point inside the triangle is constant and equal to the triangle's height.

Calculating area in two different ways is a major technique for establishing results in geometry and analysis, and the CME Project regularly reinforces this habit of mind.

Involving the Community

Our perspective is that criticism from every corner of the mathematical community is essential to our work. To implement our guiding principles, we built a team of developers, advisors, and reviewers that includes, in addition to teachers and curriculum experts, mathematicians and scientists from the research community, business, and industry. We took care to include people who were likely to have different points of view.

The CME Project had a Teacher Advisory Board (see p. v) that met regularly during development cycles. The teacher advisory board included teachers who said they will never use a text that has worked-out examples and teachers who said that students should never tackle a problem unless they are given instruction on how to solve it.

We did not go to either of these extremes, but this tension and others like it had a substantial influence on our design.

Our National Advisory Board (see page v) included people with very diverse perspectives. In addition to making for very spirited advisory board meetings, the varying perspectives we got from these advisors, consultants, and reviewers has greatly enriched our work and has made it highly likely that we hear about errors, criticisms, and points of view different from ours.

As with our monthly teacher meetings, we did not take all the advice we got, but even suggestions that were not implemented had an indirect effect on the finished product. Also, the materials went through extensive field testing. We describe some of what our field-test teachers did for us on page v.

We also invited students to advise us on the development, and to comment along the way. Students in field-test classes were generous with their feedback, and also welcoming as we observed their classes, allowing us to listen in on their conversations.

In this way, we believe that the program reflects common wisdom across the entire community.

The CME Project went through extensive field testing during the development cycle. Although research continues, early findings from an independent evaluator gave evidence of increased gains for students in CME Project Algebra 1 compared to a non-CME Project classroom.
# Mathematics Overview of the Four-Year Program

## Algebra 1

### Chapter 1  Arithmetic to Algebra

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Title</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A The Tables of Arithmetic</td>
<td>Students examine patterns in addition and multiplication tables to review the rules for addition and multiplication, and to extend the rules to negative integers. The investigation challenges students to work like mathematicians and to understand the rules they use.</td>
<td></td>
</tr>
<tr>
<td>1B The Number Line</td>
<td>Students review the basics of the number line for integers and extend ideas past the integers to include the rational and real numbers. The process of extension helps students use the number line to visualize addition and multiplication of any two numbers (not just integers), and to see that the basic rules they know carry over into these new sets of numbers.</td>
<td></td>
</tr>
<tr>
<td>1C The Algorithms of Arithmetic</td>
<td>Students reexamine in detail the algorithms they use to add, subtract, multiply, and divide to see how these algorithms use the basic rules of arithmetic, such as the commutative, associative, and distributive properties. The investigation encourages students to look at the study of algebra as not only about finding a method that works but also about understanding why that method works.</td>
<td></td>
</tr>
</tbody>
</table>

### Chapter 2  Expressions and Equations

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Title</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2A Expressions</td>
<td>Students begin to transform algorithms, such as following simple number tricks, into expressions using variables. Students define variables as placeholders that provide a shorthand for expressing the patterns they find, and for understanding why those patterns occur.</td>
<td></td>
</tr>
<tr>
<td>2B Equations</td>
<td>This investigation introduces equations to express relationships between expressions. Students solve simple equations using backtracking—a method that has students think of the equation as a series of steps applied to a number ( x ), and then undo each step in reverse order to find the initial value of ( x ). This instinctive process serves as a starting place for more formal equation solving.</td>
<td></td>
</tr>
</tbody>
</table>
Chapter 2  Expressions and Equations  (continued)

| INVESTIGATION | 2C  Solving Linear Equations | Students begin to formalize the basic methods for solving equations. The basic rules and moves of equations are those operations that they can perform without changing the solution set of the equation, such as adding the same number to each side of an equation and multiplying each side of an equation by the same nonzero number. Students also explore why these moves do not change the solutions of the equation. |

| INVESTIGATION | 2D  Word Problems | Students use the guess-check-generalize method for building equations from situations. The method involves taking several guesses at the answer, checking those guesses against the text of the problem, and keeping careful track of the steps followed to check the guesses. Finally, students guess with an arbitrary number (the variable) to build the equation. From there, they use skills from earlier in the investigation to solve the equation. |

Chapter 3  Graphs

| INVESTIGATION | 3A  Introduction to Coordinates | This investigation reintroduces the coordinate plane as students experiment with transformations of points and shapes and explore absolute value and distance. Students see how the two axes can represent different data points. They also see how graphs can show trends in data, a topic explored further throughout this chapter and in Chapter 4. |

| INVESTIGATION | 3B  Statistical Data | Students use graphs, charts, and tables to summarize and interpret data. They recognize and construct visual representations of data, including box-and-whisker plots and scatter plots. They use their interpretations to make informed conclusions about data. |

| INVESTIGATION | 3C  Equations and Their Graphs | Students learn that the graph of an equation is just another representation of the set of points that make the equation true. The investigation uses simple and complex graphs to reinforce the idea that no matter how difficult an equation is, students can use it as a point-tester to check whether a particular point lies on the graph of the equation. |

| INVESTIGATION | 3D  Basic Graphs and Translations | Students learn six basic graphs, i.e., the graphs of six basic functions. They learn connections between horizontal and vertical translations of the graphs and algebraic representations of the functions. |
### Algebra 1 (continued)

#### Chapter 4  Lines

| 4A All About Slope | The study of slope begins with the definition of slope between two points. Students test for collinearity of points by using the idea that three points are collinear if and only if the slope between each pair of them is the same. Ultimately, they prove the corollary that slope is invariant for all pairs of points on a line. They use this invariance as the point-tester to see whether some given point is or is not on a line, and eventually (in the next investigation) to develop an equation for the line itself. |
| 4B Linear Equations and Graphs | This investigation resolves the question from the previous investigation by having students use the point-tester concept to develop a general method for finding an equation of a line. This course does not emphasize any particular form of a linear equation, but rather works on the overriding principle that to graph a line, only two points on that line need to be found, and any two points will do. |
| 4C Intersections | Students solve systems of linear equations using the substitution and elimination methods. While the explanation of these two methods is basically traditional, the exposition relies on and emphasizes the basic moves and the point-tester concept. The proof that lines with the same slope are parallel introduces students to the concept of proof by contradiction. |
| 4D Applications of Lines | Students apply their work with lines to solve inequalities and estimate the line of best fit. Students explore an inequality by graphing each side of the inequality and then comparing the y-heights of the two graphs. Students find fitting lines for a data set by determining the balance point of the data and estimating the slope of the line. Students compare their lines with the actual data, calculate simple errors, and think about how to minimize those errors. |
## Chapter 5 Introduction to Functions

### 5A Functions—The Basics
This investigation introduces functions as machines defined by specialized rules—rules that assign each input exactly one output. Students make their own rules for given sets of inputs and outputs. From this foundation, they generate tables, algebraic expressions, and ultimately graphs. The lessons gradually add more formal algebra for expressing rules (such as $f(x)$ notation and the concept of domain).

### 5B Functions, Graphs, and Tables
Students fit functions to tables. First, they explore differences in successive outputs of a function and determine that constant differences imply linear functions. Next, students meet recursive rules that describe some tables. Finally, students use these recursive rules to fit exponential functions to tables with constant ratios.

### 5C Functions and Situations
Students extend their work from the end of Chapter 2 and use the guess-check-generalize method to build functions that model situations described in word problems.

## Chapter 6 Exponents and Radicals

### 6A Exponents
Following a process similar to the process from Chapter 1, students develop the basic rules of exponents, starting with positive integer exponents. Students use the rules to find sensible definitions for zero and negative exponents.

### 6B Radicals
Although most students are familiar with square roots prior to algebra 1, this investigation treats the subject more deeply. Students learn the differences between rational numbers and irrational numbers, and basic rules and conventions for calculating with square roots. The final lessons treat other radicals, such as cube roots, fourth roots, and, more generally, $n$th roots.

### 6C Exponential Expressions and Functions
Students explore exponential functions by looking at their graphs, exploring quotient tables (similar to the difference tables they saw in Chapter 5), and calculating compound interest. Students will investigate these topics further in algebra 2.
## Algebra 1 (continued)

### Chapter 7 Polynomials

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>7A The Need for Identities</strong></td>
<td>The heart of this investigation lies in factors—forming them, expanding them, comparing them, and ultimately using them in the development of the Zero Product Property. The exercises extend all of the basic moves of equations students know and has students recognize equivalent expressions and develop algebraic identities.</td>
</tr>
<tr>
<td><strong>7B Polynomials and Their Arithmetic</strong></td>
<td>This investigation introduces monomial and polynomial. Students explore the features of polynomial expressions. There is extra practice for adding and multiplying polynomials, combining like terms, and factoring out the greatest common monomial factor of a polynomial.</td>
</tr>
<tr>
<td><strong>7C Factoring to Solve: Quadratics</strong></td>
<td>Students factor quadratic expressions. Earlier lessons have previewed this work and students have reviewed and practiced it. Here students study techniques for factoring quadratic expressions in depth. They use factoring to solve quadratic equations.</td>
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</tbody>
</table>

### Chapter 8 Quadratics

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>8A The Quadratic Formula</strong></td>
<td>This investigation derives the quadratic formula through the process of completing the square for a general quadratic equation. Students develop expertise in solving quadratic equations. They develop a flexible understanding of the relationship between a quadratic equation and its roots. They write a quadratic equation so that it has specific roots.</td>
</tr>
<tr>
<td><strong>8B Quadratic Graphs and Applications</strong></td>
<td>Students develop techniques for graphing quadratic equations, paying special attention to the roots and vertex. They use these graphs to solve for maximums and minimums in word problems.</td>
</tr>
<tr>
<td><strong>8C Working With Quadratics</strong></td>
<td>Students look again at solving equations and inequalities by graphing each side of the equation or inequality as a distinct function, and comparing the graphs. They also look at advanced inequalities, such as systems of inequalities, both linear and quadratic. Finally, they revisit the idea of difference tables for quadratics.</td>
</tr>
</tbody>
</table>
### Geometry

#### Chapter 1  An Informal Introduction to Geometry

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1A Picturing and Drawing</strong></td>
<td>Students examine techniques for drawing figures and specifying critical features of a drawing. These include symmetry and connections to algebra.</td>
</tr>
<tr>
<td><strong>1B Constructing</strong></td>
<td>Students compare and contrast constructing a figure and drawing a figure. Students investigate some basic properties of triangles.</td>
</tr>
<tr>
<td><strong>1C Geometry Software</strong></td>
<td>Students investigate tools and features of geometry software by making geometric constructions. Students explore dynamic construction and find invariant measurements and relationships.</td>
</tr>
<tr>
<td><strong>1D Invariants</strong></td>
<td>Students examine algebraic invariants, including difference and ratio, and geometric invariants, including shape, concurrence, collinearity, measurement, and congruence.</td>
</tr>
</tbody>
</table>

#### Chapter 2  Congruence and Proof

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>2A The Congruence Relationship</strong></td>
<td>Students define congruence and figure out the minimum conditions they need to prove congruence for different kinds of figures, including segments, angles, and triangles. Students use basic postulates of triangle congruence to prove theorems.</td>
</tr>
<tr>
<td><strong>2B Proof and Parallel Lines</strong></td>
<td>This investigation explains the deductive method of proof. Students put it into practice with theorems about parallel and perpendicular lines. Students prove and use the theorems, along with previous results, to prove more theorems.</td>
</tr>
<tr>
<td><strong>2C Writing Proofs</strong></td>
<td>Students examine several techniques for finding a path from hypothesis to conclusion, presenting an argument, and writing formal proofs. Students evaluate and complete example proofs in addition to writing their own.</td>
</tr>
<tr>
<td><strong>2D Quadrilaterals and Their Properties</strong></td>
<td>Students define and classify quadrilaterals. They examine properties of quadrilaterals, develop conjectures, and prove theorems about quadrilaterals.</td>
</tr>
</tbody>
</table>
Geometry (continued)

Chapter 3 Dissections and Area

3A Cut and Rearrange
This investigation develops algorithms for cutting a polygon and rearranging the pieces to form a specific type of polygon. Students analyze the algorithms and prove they are valid. Students use dissection to prove the Midline Theorem.

3B Area Formulas
Given the formula for the area of a rectangle, students use dissections to derive area formulas for various other polygons including triangles, parallelograms, and trapezoids.

3C Proof by Dissection
Students prove the Pythagorean Theorem in several different ways using different dissections. Students apply the Pythagorean Theorem in different contexts and explore Pythagorean triples.

3D Measuring Solids
Students use nets to find surface areas of solids. Students use dissections to find volumes. The investigation defines pyramids and prisms, and students make sense of formulas for their surface areas and volumes. Students relate cones and cylinders to pyramids and prisms, and do likewise with formulas for the respective surface areas and volumes.

Chapter 4 Similarity

4A Scaled Copies
This investigation introduces similarity informally, using the idea of scaled drawings. Students read scaled drawings and develop their own methods to make them. They also determine how to tell whether one drawing is an accurate scaled copy of another.

4B Curved or Straight? Just Dilate!
Students construct scaled drawings using both the ratio and parallel methods. In the ratio method, they draw segments from a center of dilation to each critical point on the figure and then scale each segment from the center to find the corresponding critical point of the image. In the parallel method, students scale one critical point as in the ratio method. They then locate other points in the scaled drawing by constructing parallels to corresponding segments in the original figure.

4C The Side-Splitter Theorems
Students prove the side-splitter theorems and use them to show that the ratio and parallel methods for scaling figures are equivalent.

4D Defining Similarity
Students learn the formal definition of similarity and develop and prove tests for similar triangles.
## Chapter 5  Circles

<table>
<thead>
<tr>
<th>5A Area and Circumference</th>
<th>Students approximate areas and perimeters of curved figures using grids and lines. Students approximate the area and circumference of a circle by inscribing and circumscribing regular polygons.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5B Circles and 3.141592653589793238462643383 . . .</td>
<td>This investigation presents and students apply formulas for area and circumference of a circle.</td>
</tr>
<tr>
<td>5C Classical Results About Circles</td>
<td>Students prove results about arcs, chords, central angles, secants, tangents, and inscribed angles and polygons.</td>
</tr>
<tr>
<td>5D Geometric Probability</td>
<td>Students investigate relationships between area and probability using Monte Carlo techniques and approach a clearer understanding of area measure.</td>
</tr>
</tbody>
</table>

## Chapter 6  Using Similarity

<table>
<thead>
<tr>
<th>6A Some Uses of Similarity</th>
<th>Students use similarity in many ways, including how to determine inaccessible distances through perspective and shadows. They prove and use some classical geometric results including the Arithmetic-Geometric Mean Inequality.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6B Exploring Right Triangles</td>
<td>This investigation introduces students to the sine, cosine, and tangent ratios. Students use these ratios to determine missing side lengths and angle measures in triangles and to find areas. They also extend the Pythagorean Theorem with the Law of Cosines.</td>
</tr>
<tr>
<td>6C Volume Formulas</td>
<td>Students extend the Chapter 3 volume formula concepts for solids to proofs of the volume formulas for prisms, cylinders, pyramids, cones, and spheres using Cavalieri’s Principle.</td>
</tr>
</tbody>
</table>
Geometry (continued)

Chapter 7 Coordinates and Vectors

| INVESTIGATION |  
|---------------|---|
| **7A Transformations** | Students perform reflections with paper folding, tracing on plain paper, the coordinate plane, and geometry software. Students compose reflections to create rotations and translations. They investigate and prove the properties of these transformations. |
| **7B Geometry in the Coordinate Plane** | Students use geometric methods to develop formulas for the midpoint of a segment, the distance between two points, and the equation of a circle on the coordinate plane. They also use slope relationships of parallel and perpendicular lines to prove geometric results. They extend concepts to three-dimensional coordinate systems. |
| **7C Connections to Algebra** | The investigation introduces vectors. Students study properties of vectors and use vectors to prove geometric relationships. |

Chapter 8 Optimization

| INVESTIGATION |  
|---------------|---|
| **8A Making the Least of a Situation** | Students investigate ways to find the shortest path in various contexts. For example, they use reflection to turn piecewise linear paths into linear paths. They use numerical examples to develop intuition about situations and then produce logical arguments to defend their solutions. |
| **8B Making the Most of a Situation** | Students develop strategies to maximize various quantities, including lengths, areas, and angle measures. They see the effects of constraints by working similar problems under different conditions. |
| **8C Contour Lines** | Students read, interpret, and make contour plots. Students analyze plots showing lines of constant elevation, distance, and angle measure. Students abstract the contour idea to show lines of constant value for mathematical functions. |
| **8D Advanced Optimization** | Students study what it means to reason by continuity—in one case to find the limit of the value of one quantity while other quantities are increased or decreased as much as possible. They use this idea to find maximums and minimums in geometric contexts. They also investigate the Isoperimetric Problem—finding the maximum area that can be enclosed by a closed curve of fixed perimeter. |
## Algebra 2

### Chapter 1  Fitting Functions to Tables

| **1A Tables** | This investigation begins the algebra 2 thread of functions and fitting. Students investigate how to use constant differences and other cues to fit linear and quadratic rules to input-output tables. They use a function-modeling language on their calculators to model and experiment with functions. |
| **1B Fitting and Data** | Students begin to develop a statistical perspective, in the sense of thinking of data in terms of trends, rather than as individual points. Students investigate whether they can reasonably approximate a data set with a linear function. They study alternatives to linear trends. Students investigate relationships among mean, median, variance, and standard deviation. |
| **1C More About Recursive Models** | Students encountered recursively defined functions in Investigation 1A. Here, they investigate recursion in greater depth by analyzing a recursively defined function that determines the monthly payment on a loan. Students also investigate the factorial function—a recursively defined function with no simple closed form. |

### Chapter 2  Functions and Polynomials

| **2A About Functions** | Students develop a “functional perspective”: What is a function? Students decide from a table, graph, or equation whether a given relationship is a function. Students also revisit notation and develop precise definitions of domain and range. Students investigate the arithmetic of functions, including composition, and have initial exposure to inverse functions. |
| **2B Making it Fit** | This investigation introduces Lagrange interpolation—a method for finding the least-degree polynomial function that passes through a given set of points. This investigation is algebraically technical. As a result, students develop fluency in algebra with polynomials and the use of a computer algebra system. |
| **2C Factors, Roots, and Zeros** | Students study the Factor Theorem and the Remainder Theorem. Students learn the relationship between roots and factors of polynomials. Students divide polynomials by monic linear polynomials. |
| **2D Advanced Factoring** | Students study various polynomial forms and methods for factoring them: lumping, scaling, and grouping. Students also study rational expressions for the first time. |
# Algebra 2 (continued)

## Chapter 3  Complex Numbers

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>3A</strong> Introduction to Complex Numbers</td>
<td>Students encounter problems whose solutions require square roots of negative numbers. They extend the concept of real numbers to complex numbers in order to include these square roots. They then investigate the arithmetic of complex numbers.</td>
</tr>
<tr>
<td><strong>3B</strong> The Complex Plane</td>
<td>Students learn magnitude and direction for complex numbers, how to graph complex numbers, and how to interpret complex-number arithmetic geometrically.</td>
</tr>
<tr>
<td><strong>3C</strong> Complex Numbers, Geometry, and Algebra</td>
<td>This investigation explores more advanced topics in complex numbers. It includes in-depth investigations of magnitude and direction, roots of polynomials, and roots of unity.</td>
</tr>
</tbody>
</table>

## Chapter 4  Linear Algebra

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4A</strong> Gaussian Elimination</td>
<td>Students solve systems of linear equations. Students use matrices and Gaussian elimination as tools for this purpose.</td>
</tr>
<tr>
<td><strong>4B</strong> Matrix Algebra</td>
<td>Students solve matrix equations. Students also multiply matrices, find inverses of matrices, and evaluate dot products, with pencil and paper and with their calculators.</td>
</tr>
<tr>
<td><strong>4C</strong> Applications of Matrix Multiplication</td>
<td>Students use matrices to represent sequences of geometric transformations, model the evolution of a system over time, and analyze sequences of repeated probabilities.</td>
</tr>
</tbody>
</table>

## Chapter 5  Exponential and Logarithmic Functions

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>5A</strong> Working With Exponents</td>
<td>This investigation begins with a review of the laws of exponents, and includes zero and negative exponents. Students investigate arithmetic and geometric sequences. They use those sequences to extend the laws of exponents to include rational exponents.</td>
</tr>
<tr>
<td><strong>5B</strong> Exponential Functions</td>
<td>Students explore exponential functions via tables and graphs.</td>
</tr>
<tr>
<td><strong>5C</strong> Logarithmic Functions</td>
<td>Students learn what logarithms are, how to work with them, and how to graph logarithmic functions. Students also learn about logarithmic scales.</td>
</tr>
</tbody>
</table>
Chapter 6  Graphs and Transformations

<table>
<thead>
<tr>
<th>INVESTIGATION</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>6A Transforming Basic Graphs</td>
<td>Students enlarge their tool kit of basic graphs to include circles and the basic cubic curve. They investigate connections between translations and stretches of graphs and the algebraic representations of the graphs.</td>
</tr>
<tr>
<td>6B Affine Transformations</td>
<td>Students investigate the algebra of functions of the form $x \rightarrow ax + b$. They apply this algebra to completing the square for quadratic equations and to reducing cubic equations to three simple forms.</td>
</tr>
<tr>
<td>6C Graphing Using Affine Transformations</td>
<td>Students study an alternative way to understand the effects of translations and dilations on graphs—instead of transforming the graphs they transform the axes.</td>
</tr>
</tbody>
</table>

Chapter 7  Sequences and Series

<table>
<thead>
<tr>
<th>INVESTIGATION</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>7A The Need to Sum</td>
<td>Students investigate how to sum integers.</td>
</tr>
<tr>
<td>7B Sum Identities</td>
<td>Students investigate definite and indefinite sums. They develop the formula for the sum of the first $n$ integers and sums of powers of a base.</td>
</tr>
<tr>
<td>7C Arithmetic and Geometric Sequences and Series</td>
<td>Students study identities for sums of finite arithmetic and geometric series and for convergent infinite geometric series. The emphasis here is on the “linearity of summation.”</td>
</tr>
<tr>
<td>7D Pascal’s Triangle and the Binomial Theorem</td>
<td>Students spend time investigating both Pascal’s Triangle and the Binomial Theorem.</td>
</tr>
</tbody>
</table>

Chapter 8  Introduction to Trigonometry

<table>
<thead>
<tr>
<th>INVESTIGATION</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>8A Trigonometric Functions</td>
<td>Students learn the definitions of sine, cosine, and tangent as they apply to angles between 0 and 360 degrees. Students solve simple equations involving trigonometric functions.</td>
</tr>
<tr>
<td>8B Graphs of Trigonometric Functions</td>
<td>Students graph sine, cosine, and tangent functions. They prove some basic trigonometric identities, including the Pythagorean Identity.</td>
</tr>
<tr>
<td>8C Applications to Triangles</td>
<td>Students investigate the area of a triangle from a trigonometric perspective and look at a geometric form of angle addition formulas. Students learn the Law of Sines and the Law of Cosines.</td>
</tr>
</tbody>
</table>
Precalculus

Chapter 1 Analyzing Trigonometric Functions

1A The Cosine and Sine Functions
This investigation introduces radian measure as the length of an arc of the unit circle. The text is careful not to equate degree and radian measurement (as one would equate inch and centimeter measurement). Students work on the customary problems of finding corresponding degree and radian measures, but a goal is to enforce the concept that a radian is a distance measurement, not another way to measure angles. Students also review the graphs of the cosine and sine functions. They develop an understanding of a periodic function. They solve simple equations that involve cosine and sine.

1B Other Trigonometric Functions
Students complete the list of trigonometric functions, learning tangent and the three reciprocal functions, secant, cosecant, and cotangent. They review the graphs of these functions, learn how each function relates to the unit circle, and work with some basic identities. They also review the definitions of inverse function and one-to-one. They learn how to restrict the domain of cosine, sine, and tangent in order to define their inverse functions.

1C Sinusoidal Functions and Their Graphs
This investigation emphasizes the idea that trigonometric functions \( f(x) = \sin x \) and \( f(x) = \cos x \) are basic functions, and their graphs are basic graphs. The sinusoidal functions emerge from the same transformations of basic graphs that students have studied beginning in algebra 1. The lessons de-emphasize the memorization of formulas related to coefficients. Instead, concrete examples motivate amplitude and phase shift.

Chapter 2 Complex Numbers and Trigonometry

2A Graphing Complex Numbers
Students explore polar form as a way to represent complex numbers. The exercises lead up to the multiplication law for complex numbers, which highlights the connection between complex numbers and transformational geometry: You can dilate and rotate a vector by performing one multiplication.

2B Trigonometric Identities
Students explore ways to build and prove trigonometric formulas and identities. Students learn that by using complex arithmetic, they can easily prove trigonometric facts that are quite difficult to show using only algebra of the real numbers and geometry of the real plane.
### Chapter 2  Complex Numbers and Trigonometry  (continued)

<table>
<thead>
<tr>
<th>2C</th>
<th>De Moivre’s Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Students study the roots of unity. This study is especially suited for showing the connections among algebra, geometry, and analysis. Representing complex numbers as trigonometric functions allows students to use analytic methods for solving algebraic problems (and vice versa). Connecting the roots of $x^n - 1 = 0$ and the vertices of the regular $n$-gon allows students to use the power of algebra for solving geometric problems and to use geometric methods for algebraic insights. Calculating with “cyclotomic integers” (complex numbers that are linear combinations of $n$th roots of unity with integer coefficients) students see once again how they can model algebraic systems with polynomials and computer algebra systems.</td>
</tr>
</tbody>
</table>

### Chapter 3  Analysis of Functions

<table>
<thead>
<tr>
<th>3A</th>
<th>Polynomial Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Students look at the graphs of polynomial functions and develop a general description of the properties of these functions, especially for polynomials of degrees 3 and 4, including their behaviors for values of $x$ that have extremely large absolute values. Students develop both a formal and an informal understanding of what continuity means. Through key theorems, including the Intermediate Value Theorem, students draw a conclusion about the number of roots possible for a polynomial function of a given degree. Students calculate the slope of the line tangent to the graph of a polynomial function at a point. They do so by finding the limit of the secant slopes as one of two points that determines the secant lines approaches the other. They also use iterated long division, the method of undetermined coefficients, and technology to write the Taylor expansion for a general polynomial function about a point. They use this expansion to find an equation for the tangent to the polynomial graph at that point. Along the way, they develop an informal understanding of what the slope of this tangent reveals about the polynomial function.</td>
</tr>
</tbody>
</table>
Precalculus (continued)

Chapter 3  Analysis of Functions (continued)

3B  Rational Functions

Students analyze the graphs of rational functions. They develop a general description of the properties of rational-function graphs, including the types of discontinuities that can occur and the behavior of these graphs for values of \( x \) that have extremely large absolute values. Students use the fact that these graphs are continuous at points where the function is defined to find an equation for the slope of the tangent to the graph at such a point. They do this in much the same way as they did for polynomial functions.

3C  Exponential and Logarithmic Functions

Investigation 3C applies the analysis of graphs to exponential and logarithmic functions. Students define the number \( e \) through an analysis of compound interest. Students see that as interest is compounded more frequently, the interest yield is greater, but that there is a limit to how large it can get. This leads to a definition of continuously compounded interest and a limit definition of \( e \). Students also learn a factorial definition of \( e \) and show that the two definitions are equivalent. Students learn of the function \( f(x) = \ln x \). They write any exponential function in terms of \( e \) and a general logarithmic function in terms of the natural logarithm. Students also analyze the functions \( f(x) = e^x \) and \( f(x) = \ln x \) as in the first two investigations, finding the equation for the line tangent to the graph at a point.

Chapter 4  Combinatorics

4A  Learning to Count

Students begin their work in combinatorics by diving into fairly challenging combinatorial problems before they see formal strategies for solving those problems. At the end of the investigation, they see three strategies: the box method, building tree diagrams, and solving a simpler problem.

4B  Permutations and Combinations

This investigation provides students with a formal introduction to permutations and combinations.

4C  Making Connections

Students revisit Pascal’s triangle with the goal of making sense of connected ideas.
## Chapter 5  Functions and Tables

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>5A A New Method of Proof</strong></td>
<td>Students review defining functions recursively and with closed forms. They then focus on when two such functions might agree. Students use mathematical induction to justify that two such rules agree for any integer input.</td>
</tr>
<tr>
<td><strong>5B Fitting Functions to Tables</strong></td>
<td>Students explore difference tables and learn connections between the construction of difference tables and the construction of Pascal's Triangle. This leads to Newton's Difference Formula, a method students can use to find a function that fits a table.</td>
</tr>
<tr>
<td><strong>5C Closed-Form and Recursive Definitions</strong></td>
<td>Students explore different rules for defining functions recursively. They find a closed form for a function defined recursively. Students also look at closed-form function definitions for investment situations and for Fibonacci numbers, among other examples.</td>
</tr>
</tbody>
</table>

## Chapter 6  Analytic Geometry

<table>
<thead>
<tr>
<th>Investigation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>6A Coordinate Geometry</strong></td>
<td>Students make more connections between algebra and geometry by using algebraic techniques to prove geometric results. However, instead of demonstrating specific instances of geometric theorems through coordinates, students choose strategic coordinate systems to construct general proofs of geometric results.</td>
</tr>
<tr>
<td><strong>6B Conic Sections</strong></td>
<td>This investigation explores conic sections from different perspectives. Students see these curves as the intersections of a plane with an infinite double cone, as collections of points with specific distance relationships to points and lines, and as graphs of equations in two variables. They make connections among these different representations as they compare and contrast the curves and their equations.</td>
</tr>
<tr>
<td><strong>6C Vector Algebra and Geometry</strong></td>
<td>Students use matrices to operate on vectors, which are ordered pairs in the plane or ordered triples in three-dimensional space. They use vector methods to connect to geometry. As students come to visualize the geometric results of algebraic operations on vectors, they learn more about the properties of those operations. Through vector methods, they arrive at completely new ways to think about geometric proofs in the coordinate plane. Vector methods are especially valuable because they translate much more easily to three-dimensional space than do traditional coordinate methods.</td>
</tr>
</tbody>
</table>
### Chapter 7  Probability and Statistics

<table>
<thead>
<tr>
<th>INVESTIGATION</th>
<th>7A Probability and Polynomials</th>
<th>Students formalize the language that describes probability experiments. They calculate and make sense of expected value as a statistical measure of these experiments. They make and use generating polynomials to calculate frequencies and probabilities.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7B Expectation and Variation</td>
<td>This investigation has students calculate expected value, variance, and standard deviation, and draw conclusions about the algebraic structure of these functions. Students also explore Bernoulli trials and see how they can simplify formulas for expected value and variance in this restricted environment.</td>
</tr>
<tr>
<td></td>
<td>7C The Normal Distribution</td>
<td>This investigation gives students a chance to develop an informal understanding of the Central Limit Theorem. By looking at probability distributions for some very different probability experiments, students see that as the number of trials of the experiment increases the distributions come to resemble each other more and more. Eventually, they all come to resemble the bell-shaped curve of the normal distribution. Students use this distribution to approximate one of their own distributions and calculate with it.</td>
</tr>
</tbody>
</table>

### Chapter 8  Ideas of Calculus

<table>
<thead>
<tr>
<th>INVESTIGATION</th>
<th>8A Finding Areas of Shapes</th>
<th>Students look at the areas of familiar shapes and then move on to think about how to find the areas of irregular or curved shapes.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8B Finding Areas Under Curves</td>
<td>Students look at areas under the graphs of $y = x^m$ between $x = a$ and $x = b$ for any positive integer $m$.</td>
</tr>
<tr>
<td></td>
<td>8C A Function Emerges</td>
<td>Students extend what they have learned and look at the area under the graph of $y = \frac{1}{2}$ between $x = 1$ and $x = a$.</td>
</tr>
</tbody>
</table>
Implementing the CME Project
Mathematical Approaches in Algebra 1

Beginning With Arithmetic

CME Project Algebra 1 opens with an examination of arithmetic to introduce students to algebraic thinking. Students explore addition and multiplication tables, finding patterns and making conjectures about the basic rules of arithmetic for whole numbers.

They extend arithmetic to negative numbers while making sure the rules remain true.

They then extend the number system from the integers \( \mathbb{Z} \) to the rational numbers \( \mathbb{Q} \) and the real numbers \( \mathbb{R} \), using the traditional correspondence between points on a number line and real numbers. Finally, students examine the basic methods for the integers

They extend arithmetic to negative numbers in a meaningful way.

In the multiplication table, \( 0 \cdot 0 \) is at the origin and the rest of the traditional table is in the first quadrant. This orientation is convenient for patterning purposes. For example, when you want to look at products of pairs with a constant sum, you inspect the products by moving along a diagonal line.

Patterns in the tables below make it clear to students how to extend the operations to negative numbers in a meaningful way.

There are many possible ways to extend the tables, but there is only one way that preserves the structure of the positive integers.

By studying the arithmetic tables, students see numbers in several different ways:

- They start to see relations between numbers: the addition table, for example, highlights some simple linear equations (such as \( x + y = 12 \)).
- They get their hands dirty exploring these tables, so when they want to talk about general patterns they see, they have the algebraic language they need.
• They start to think of arithmetic operations as two-variable functions that accept two numbers for inputs and return one output. Although they do not use that language here, these tables set students up to work with two-variable functions later in the program.

Students may be inclined to rush through finding patterns in these tables, because the arithmetic is so familiar. There are reasons, however, to take some time here.

For example, students might look in the addition table at all the numbers with a sum of 12. If they then scan the same diagonal in the multiplication table, they will see the possible products of numbers with a sum of 12. They may notice, in particular, the location where the product is greatest (the “center” of the diagonal, at 6 • 6).

Having thought about these kinds of arithmetic combinations will prove useful when students factor quadratics.

Low Threshold, High Ceiling

These arithmetic tables are an example of “low-threshold, high-ceiling” explorations. There is plenty of mathematics to be found in them. Mathematicians and mathematics teachers alike have gotten wrapped up in looking at patterns in these tables!

Some of the patterns students will find are more obvious. For example, in the addition table, if you look at two adjacent numbers in one row, the one on the right is one greater than the one on the left.

Some patterns are less obvious. For example, in the multiplication table, draw a rectangle with sides parallel to the axes. The products of the numbers at opposite corners are equal.
There are, in fact, several sensible ways to continue the pattern. For example, moving to the left from Quadrant I to Quadrant II, the numbers might decrease and then increase. But those extensions do not preserve the rules of arithmetic.

In this way, the desire to keep things like the distributive law working in the extended domain imposes on students the fact that \((-2)(-3) = 6\). Lurking in the background here are proofs of why things must behave the way they do, but *Algebra 1* does not get that formal.

If you think about it, those formal proofs do exactly what this approach does—they show what \((-a)(-b)\) must be, not for some arbitrary reason, but because mathematicians want to calculate in predictable ways.

This approach—defining things so that they behave in a desired way—is used to define, for example, rational exponents, complex numbers, matrix multiplication, and irrational exponents. It is important that students understand that definitions in mathematics are made by people, from their experiences, and for very specific purposes.

**The Number-Line Model**

Students also use the number line to extend their knowledge to real numbers. The number line equates measurement, specifically distance, to number.

By looking at the number line and appreciating that any point on the number line corresponds to a number, students discover that there are locations on that number line that do not correspond to integers. That realization leads to the definitions of the rational and irrational numbers.

Students then look at addition on the number line. The representation here is a subtle introduction to vectors—in this case, one-dimensional vectors. Students see that, for addition to work, they must align the “arrows” end to end—an important requirement when working with vectors of two dimensions (and greater)!

This model of addition can apply to any real numbers, while previous models only work with rational numbers.

**Development of Multiplication Models**

Students encounter expansion boxes in the context of arithmetic at the end of Chapter 1 of CME Project *Algebra 1*. They begin to understand why traditional multiplication algorithms work by representing the multiplication process with an expansion box.

To represent the equation \(327 \times 6 = 1962\), students make an expansion box by writing the place-value parts for 327 in the first column and the place-value parts for 6 in the first row. Each entry in the table is the product of its row and column.

<table>
<thead>
<tr>
<th></th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>1800</td>
</tr>
<tr>
<td>20</td>
<td>120</td>
</tr>
<tr>
<td>7</td>
<td>42</td>
</tr>
</tbody>
</table>

Students then carry out the addition.

\[1800 + 120 + 42 = 1962\]

Most students do not need an expansion box to carry out a calculation like this. However, the CME Project uses expansion boxes throughout the algebra courses as a way of multiplying polynomials. A simple example of this is an expansion box to multiply \(7(3x - 4)\).

<table>
<thead>
<tr>
<th></th>
<th>3x</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>21x</td>
<td>-28</td>
</tr>
</tbody>
</table>

So \(7(3x - 4) = 21x + (-28) = 21x - 28\).
Students can use this for representing other products. For instance, they can represent the product \((3x - 2)(x^2 - 4x + 4)\) as

\[
\begin{array}{c|cc}
  x^2 & -4x & 4 \\
\hline
  3x & & \\
-2 & & \\
\end{array}
\]

and then carry out the multiplication and addition.

By organizing multiplication in this way, students can focus on how the distributive property works (because you find this product with repeated applications of the distributive property) and not on worrying about confusion with signs.

As the course progresses, and at their own pace, students move away from this representation. In CME Project field tests, teachers often described students as “grasping” these methods. Students can generalize these methods to any polynomials they encounter.

Field tests also suggest that even as students outgrow the representation, they retain the underlying principles governing the model.

**Backtracking**

In CME Project *Algebra 1*, students begin to make the connection between finding solutions to equations and finding inverses for functions.

Before any of the formalisms about solving linear equations, students use a method called backtracking to solve equations such as \(\frac{3(x + 2) - 7}{5} = 4\).

The course presents such equations with descriptions along the lines of

I took a number, added 2, tripled the result, subtracted 7, and divided the result by 5. I ended with 4. What number did I start with?

The left side of the equation becomes a description of an algorithm, a function defined by a sequence of arithmetic calculations. Students model this algorithm in many ways; one useful representation is as a machine network.

\[
\begin{array}{ccc}
  x & +2 & \rightarrow \\
  & \times3 & \rightarrow \\
  & -7 & \rightarrow \\
  & +5 & \rightarrow ?
\end{array}
\]

a machine model for \(x \rightarrow \frac{3(x + 2) - 7}{5}\)

Students practice running several inputs through the network. Then *Algebra 1* asks them to “pull back” an output to get to the corresponding input.

To do this, they perform inverse steps in reverse order, finding a solution of the equation. In fact, an extension exercise could ask them to build a network that gives the output value \(x\) for any input value \(k\) for the equation \(\frac{3(x + 2) - 7}{5} = k\).

**Algebra Word Problems**

The difficulties that high school students have with word problems are legendary. The quintessential word problem (“Mary is 10 years older than her brother was 5 years ago . . .”) is the topic of cartoons and jokes.

Teachers have devoted a great deal of effort to exposing the roots of the difficulties people have with word problems. Two very common perceptions are that students have difficulty with word problems because

- they have a general difficulty reading
- they are often not familiar with the contexts described in the problems

But an analysis by some middle and high school teachers in Woburn, Massachusetts, showed that there is more to it. They observed that prealgebra students who understand the connection between rate, time, and distance have very little difficulty with the following problem.

Mary drives from Boston to Chicago, a trip of 1000 miles. Then she drives back
to Boston. If she travels at an average rate of 60 mi/h on the way to Chicago and 50 mi/h on the way back, how many hours does her trip take?

But the following problem is baffling to many of the same students a year later in algebra class.

Mary drives from Boston to Chicago and then drives back to Boston. She travels at an average rate of 60 mi/h on the way to Chicago and 50 mi/h on the way back. If the total trip takes 36 hours, how far is it from Boston to Chicago?

This analysis led to an effective method—guess-check-generalize—for solving this kind of problem. Do not confuse guess-check-generalize with the well-known guess and check strategy for finding solutions or approximate solutions to numerical problems.

Here is how guess-check-generalize worked for one student, Kyle, for the second problem above.

Kyle took a guess of 2000 miles.

He saw that if Chicago were 2000 miles from Boston, the round trip would take $73\frac{1}{3}$ hours. So his answer was not right, but that was okay.

The teacher asked Kyle to be explicit about what he did to check his guess. He was not sure, so he took another guess:

The purpose of these guesses is not to stumble on a right answer. The important part is for students to focus on the steps they take to check the guess.

After a couple of rounds of this, Kyle was able to articulate, “You take the guess, divide it by 60, then divide it by 50. Add your answers and see if you get 36.”

He then wrote down a generic “guess-checker” this way:

And this guess-checker gives the equation that models the problem.
Equations and Graphs

Many high school students do not understand that they can test a point to see if it is on the graph of an equation by seeing if its coordinates satisfy the equation. Equations are, for students who do understand, a kind of code from which they can read off information that allows them to produce a graph.

The CME Project Algebra 1 approach to this phenomenon is to provide students with opportunities to connect equations and graphs without elaborate formalisms. Students use the idea that the equation of a graph is the point-tester for the graph. You use the point-tester to tell you whether a point is on the graph by checking some numerical fact(s) about its coordinates.

For example, an exercise asks students to find an equation of the horizontal line that passes through (5, 1). They typically have no trouble drawing the line. And, when given several points, they usually have no trouble explaining why each is on or off the line: To see if a point is on the line, check to see if its y-coordinate is 1. So, the point-tester equation is \( y = 1 \).

Another example: Find an equation of the line with a graph that bisects Quadrants I and III. To solve this, students can check to see whether the coordinates of a point on the line are the same. The equation is thus \( y = x \).

These are simple examples, but they reinforce the meaning of the correspondence between equations and their graphs.

And the point-tester idea works well for more complex equations and their graphs. It is an idea that runs throughout the program. The CME Project develops the idea of point-testers for graphs. Then it moves into the example of lines, so students do general work with graphing before they graph linear functions.

Slope and Equations of Lines

In CME Project Algebra 1, the point-tester idea helps students find equations for lines. The method involves a somewhat unorthodox approach to slope.

Field testing shows that an approach using the slope of a line places undue cognitive demands on students. In this approach, you ask students to think about a number (slope) that is an invariant of an infinite geometric object (the line). This is difficult for at least two reasons.

- The invariant is not part of the geometric object itself. It is a numerical quantity derived from the geometry of the line. Indeed, the slope of a line is an example of the derivative that students will study in calculus.
- The invariant is derived via a calculation that seems at first glance to depend on a choice of two points on the line.

In CME Project Algebra 1, development starts with the more concrete idea of slope between two points, a number that can be calculated directly from coordinates. This approach to equations for lines synthesizes slope with the point-tester idea. Algebra 1 makes the following explicit assumption (proved in CME Project Geometry; the proof requires results about similar triangles).

**Assumption** Three points \( A, B, \) and \( C \) lie on the same line if and only if

\[
m(A, B) = m(B, C)
\]

The CME Project uses the notation \( m(A, B) \) for the slope between \( A \) and \( B \).

Suppose you are given two points, say \( A = (3, -1) \) and \( B = (5, 3) \). What is the equation of the line that contains \( A \) and \( B \)?

Students develop the habit of checking several points to see if they are on the line, keeping track of their steps. The book first gives them some points to check, say \( X = (7, 6), P = (1, -5), \) and \( Q = (9.5, 10.5) \).

In each case, the students check the slope between the point to be tested and, say, \( B \). They want to see if it is equal to \( m(A, B) \).
that is, 2. (This method for determining the equation of a line is another example of the guess-check-generalize habit.)

The generic check is that the slope between \((x, y)\) and \((5, 3)\) should equal 2. So the point-tester is

\[
\frac{y - 3}{x - 5} = 2
\]

which students transform into a linear equation in \(x\) and \(y\),

\[
y - 3 = 2(x - 5)
\]

You can transform this equation into various other forms. But *Algebra 1* is careful not to focus on naming the forms or on switching from one form to another.

The goal is to have students use whatever version of an equation serves the purpose at hand.

After this foundation becomes solid, students proceed to sketching lines from their equations and finding equations for given lines.

Annette Roskam, a field-test teacher in Rice Lake, Wisconsin, wrote

I got so excited, the students were very concerned! I've never had a student come up with how to use slope-intercept form to find the equation of a line. Before, all I've gotten were blank stares!

**Functions as Machines**

The CME Project *Algebra 1* treatment of functions is fairly informal. The book introduces functions as machines.

The domain of a function is the set of possible inputs into the machine. The range is the set of all possible outputs from it.

Students are used to the machine model from their work solving equations in Chapter 2, so they already know how functions behave. They know that some machines are invertible and some are not. So *Algebra 1* uses a familiar model to help students understand domain, range, and the basics of functions.

The CME Project does not provide formal definitions related to functions until *Algebra 2*. (For an outline of how the CME Project develops the function concept across courses, see page 78.)

To reinforce the notions of function as machine, and domain as the set of allowable inputs, *Algebra 1* asks students to build computational models of functions in their function-modeling language.

For example, a model of \(x \to \frac{1}{x^2 - 1}\) might be

Define \(f(x) = \frac{1}{x^2 - 1}\).

Students can now use their model of \(f\), evaluating it for particular values of \(x\), graphing it, and looking for numbers that cause the model to produce an error. These numbers are excluded from the domain.

**Extension: Zero and Negative Exponents**

Most students come from middle school having some experience with positive integer exponents. They know that \(2^3\) means \(2 \times 2 \times 2\) and, more generally, if \(n\) is a positive integer, they know that

\[2^n = 2 \times 2 \times \ldots \times 2\]

\(n\) times

CME Project *Algebra 1* uses this definition to develop rules for arithmetic with positive integer exponents:

1. \(a^m \cdot a^n = a^{m+n}\)
2. \(\frac{a^m}{a^n} = a^{m-n}\)
3. \((a^m)^n = a^{mn}\)

Requiring that these definitions extend to all integers leads to some new definitions.

- \(3^0\) has to be 1 if Rule 1 extends to 0.
  
  \[3^5 \cdot 3^0 = 3^{5+0} = 3^5\]
  
  Since the only number you can multiply \(3^5\) by to get 1 is 1, \(3^0\) must be 1.
Similarly, $3^{-1}$ has to be $\frac{1}{3}$ if Rule 1 extends to negative integers.

$$3^{-1} \cdot 3^1 = 3^{-1+1} = 3^0 = 1$$

Since the only number you can multiply by $3^1$ to get 1 is $\frac{1}{3}$, $3^{-1}$ must be $\frac{1}{3}$. In the same way, $3^{-2} \cdot 3^2 = 1$, so $3^{-2}$ must be $\frac{1}{3^2}$.

If the rules extend to fractional exponents, $3^{\frac{1}{2}}$ would have to be a number with square root of 3 (by Rule 3),

$$(3^{\frac{1}{2}})^2 = 3^{\frac{1}{2} \cdot 2} = 3^{1} = 3$$

There are two choices here, $\sqrt{3}$ and $-\sqrt{3}$, and mathematicians have made the choice that $3^{\frac{1}{2}} = \sqrt{3}$.

Similarly, $3^\frac{1}{8}$ is a number with 8th power of 3, so mathematicians define it to be $\sqrt[8]{3^2}$. Later in the program, students see that this is the same as $(\sqrt[8]{3})^2$.

It is possible to extend the meaning of exponents to all integers (in CME Project Algebra 1) and all rational numbers (in CME Project Algebra 2) by requiring rules 1, 2, and 3 to be valid in those domains. But what do basic rules require for irrational exponents? What must $3^{\sqrt{2}}$ mean?

For this, the CME Project uses extension by continuity. In Algebra 2, students see that the graph of $y = 3^x$ is actually full of holes. Even though it looks smooth, there is a hole over each irrational number. If you fill in the holes, you get the definition of $3^x$ for irrational $x$ values. Put another way, $3^{\sqrt{2}}$ is the number approached by the sequence

$$3^{1.4}, 3^{1.41}, 3^{1.414}, 3^{1.4142}, 3^{1.41421}, \ldots,$$

where the sequence of rational exponents has $\sqrt{2}$ as a limit. (Again, this is all informal and intuitive, although courses in post-calculus analysis can demonstrate the result rigorously.)

**Polynomial Algebra and Formal Calculations**

One way to use polynomials as modeling tools is to look at polynomials as objects that define functions. In this way, you can model gravity with a quadratic polynomial. You can model volume with a cubic polynomial. When you view polynomials as functions, you think of the $x$ as a variable, a generic element of some replacement set.

There is another use of polynomials in which the $x$ is an indeterminate. In this view of a polynomial, the letters are just placeholders (rather than “valueholders”). What is really important are the operations between the letters.

Two common activities in school algebra, simplifying and graphing, illustrate the difference. When students simplify polynomials, they are thinking of them as formal objects. The fact that

$$x^2 - 1 = (x - 1)(x + 1)$$

comes from the fact that if you expand the right side following the rules of algebra, you end up with the left side. Graphing, on the other hand, requires that students think of substituting values for $x$, or that they imagine $x$ sweeping across some domain, producing points on a graph.

Both of these points of view are important and not completely disconnected. The CME Project wants students to be able to move between them, sometimes in the same problem.

In beginning algebra, for example, it is often helpful to ignore the fact that the letters stand for numbers. There are also times when the form of a calculation is more important and you never think of replacing the letters with values.

For example, at several points the program asks students to investigate the distribution of possible sums when three (or more) number cubes are thrown. The book then
asks them to find the coefficient of $x^9$ when they expand $(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$.

The object here is not to perform the expansion by hand or machine, but to reason where an $x^9$ term can occur when students multiply out the expression.

The formal calculation here models the counting problem for three number cubes, providing another way algebra can be used as a modeling tool. Such reasoning requires a kind of “decontextualization,” a formal approach to polynomial calculations that is important enough to deserve increased attention in the later years of high school.

The lines between form and value are not clear. By replacing $x$ with $-1$ in the unexpanded form, students can show that the number of outcomes with an odd sum is the same as the number of even outcomes when any number of number cubes are thrown.

Factoring

The CME Project’s approach to factoring polynomials provides students with enough practice that they can reason about the development of the various factoring methods. Two worth noting are the following.

- In *Algebra 1*, students factor quadratic polynomials with leading coefficient 1 (monic polynomials), using the usual search for sums and products. In *Algebra 2*, the correspondence between the roots of a polynomial equation and the linear factors of the polynomial helps students reduce the case of nonmonic factoring to the monic one by a method that essentially scales the roots of the polynomial. (See page 60 for more details about this scaling method.)

- The scaling method is a special case of substituting, or “lumping together,” to reduce complex factoring problems to simpler ones. For example, both $25x^2 + 15x - 4 = (5x)^2 + 3(5x) - 4$ and $x^4 + 3x^2 - 4 = (x^2)^2 + 3(x^2) - 4$ have the form $z^2 + 3z - 4$, which you can easily factor.

This method is revisited in CME Project *Algebra 2 and Precalculus*. When students face equations such as $5\sin x + 7 = 10$, they recognize that they can first solve $5z + 7 = 10$, which they already know how to do.

**Factoring Quadratics**

Factoring a quadratic polynomial $x^2 + px + q$ (with leading coefficient 1) comes down to finding numbers $\alpha$ and $\beta$ such that

$$\alpha + \beta = p \text{ and } \alpha \beta = q$$

So, to start students thinking about factoring, *Algebra 1* emphasizes sum-and-product problems. It places a special emphasis on squares, using the multiplication table from Chapter 1.

It asks students to solve problems such as

Find two numbers with the indicated sum and product.

- sum 20, product 100
- sum 20, product 99
- sum 20, product 96
- sum 20, product 75
- sum 20, product 36
- sum 20, product $-125$
- sum 20, product 47
- sum 20, product $n$

(The last two items on this list are much more difficult than any of the previous ones!)

The sum-and-product approach is useful for the following reasons.

- First-year algebra teachers report that students understand the method and can get up to speed with it in one or two class periods.
- Students can generalize this process. When students can describe the steps for specific numbers, they then can find the pair of numbers with sum $p$ and product $q$. At that point, students are very close to proving the quadratic formula.
There is a payoff here for having arranged the addition and multiplication tables as in Chapter 1. To factor \( x^2 - 12x + 35 \), students need to find numbers \( a \) and \( b \) with sum 12 and product 35. Using the multiplication table, they want to locate all pairs \( (a, b) \) such that \( a + b = 12 \) and \( ab = 35 \). The pairs with sum 12 in the table lie along the line with equation \( x + y = 12 \).

**Statistics in Algebra 1**

One of the goals in the CME Project is to connect basic statistical concepts to the rest of the curriculum. *Algebra 1* introduces students to basic statistics of one-variable data sets (mean, median, and mode), to geometric ways of representing data (histograms and stem-and-leaf plots), and to the effects of algebraic transformations on data sets.

For example, one exercise from Chapter 3 asks:

Consider this set of data.

20, 56, 45, 60, 77, 82, 32

Describe how each data set below is obtained from the one above. Describe how the mean and median change.

- a. 19, 55, 44, 59, 76, 81, 31
- b. 30, 66, 55, 70, 87, 92, 42

The CME Project revisits his theme throughout the program. (See page 66 of this guide, for example.)

*Algebra 1* includes an introduction to the line of best fit for a set of data. One of the goals is to show exactly what the line of best fit minimizes. While *Algebra 2* develops the formula for regression lines, *Algebra 1* students develop an understanding of what a line of best fit is. They begin to see some of its properties—for example, that the best fit line always contains the centroid of the data.
Implementing the CME Project

Mathematical Approaches in Geometry

Approaching Proof as a Mathematician

A main focus of CME Project Geometry is on logical reasoning and proof. Proof is a crucial activity in mathematics. It is how people discover and communicate new mathematics. For students, puzzling out existing proofs and developing their own are important ways to acquire deep mathematical understanding.

In earlier work in mathematics, students often had to explain their reasoning or show how they got their answers. In geometry, these explanations and demonstrations really come to the fore.

Experience Before Formalization

Each CME Project course expects students to experiment with new concepts, grapple with new questions, and work with new ideas before they see a formal presentation or example. This principle is particularly apparent in CME Project Geometry where the entire first chapter is an exploration and a preview of the rest of the course.

Geometric thinking consists of ways to approach problems, often visually, paying attention to size, shape, orientation, location, and relationships among these qualities. Geometry calls on a set of mathematical habits of mind that differ a bit from those in algebra (such as reasoning by continuity) with different recording techniques and processes.

Rather than diving into the formal definitions, theorems, and proofs of geometry, and leaving students to figure out geometric thinking on their own, CME Project Geometry starts by giving students a chance to make the transition from algebraic calculations to the deductive reasoning characteristic of geometry.

In Chapter 1, “An Informal Introduction to Geometry,” students do not work to master the content of the problems with which they struggle, but rather they learn how to struggle and stick with an unfamiliar problem as they become accustomed to the content of the course.

The main goals of this initial chapter are to:

- assess students’ geometry background knowledge
- develop a classroom culture of inquiry and collegiality

Students will puzzle over unfamiliar problems, ask questions, and share ideas. They will record and refine conjectures. They will access their background knowledge of geometry and remember previously learned vocabulary and theorems. They will learn techniques for sketches and constructions that will allow them to represent and interpret information in diagrams drawn by hand or with technology. They will begin to experiment thoughtfully, controlling some parameters while allowing others to vary. They will identify relationships and invariants in dynamic situations.

In fact, the Chapter 1 exploratory activities are suggestive of how investigations in mathematics begin in the real world.

In future chapters, students will formalize and prove conjectures that they develop in this initial experiential phase.

That does not mean that students are done with experimentation, though. Throughout the course when new concepts are introduced or new connections are made, students will start by venturing out on their own, just as mathematicians experiment when beginning to develop conjectures and proofs.
Conception vs. Presentation
CME Project Geometry makes a distinction between how a student conceives a proof—the process of coming up with the logical chain of ideas leading from hypothesis to conclusion—and how a student presents a proof. Although students see and use different styles of presentation, including two-column proofs, outline proofs, and paragraphs, Geometry takes time and places real emphasis on techniques that help students discover and refine their final deductive argument.

Tools to develop proof include

- **Visual scan**—A sketch of the situation shows critical features including given information. Students look for characteristics that will allow them to apply a known theorem in order to deduce more.
- **Flow chart**—Working from the given information, students think of as many results as they can show from what they know. Each result adds to what they know and may allow them to make more conclusions. This method can lead to more than one proof of the same result, or even a list of all the possible results that students can prove from the given information.
- **Reverse list**—This technique works backwards from the conclusion. Students ask, “What do I need?” and “What could I use to prove that?” In using the reverse list, students become more aware of the conclusions of theorems that they use and which theorem is likely to be useful in a particular situation.

All of these techniques work in synergy to help students develop a proof. Students may start out with a sketch. They may move to a flowchart when they find it difficult to keep track of all the different possible pathways to the conclusion. If they get stuck, they might use a reverse list, trying to work backward to reach one of the branches in their flowchart.

However, two of the most important ways in which students learn to devise proofs are through doing it themselves and observing others model the process.

The more students are able to watch you or some of their classmates work through a difficult piece of reasoning, with all the messy backtracking and starting over that such reasoning often entails, the more they will be able to tolerate their own struggles. And, the more students are able to persist with difficult proofs and experience the thrill of personally completing one of these complex chains of conclusions, the more confidence and endurance they will have for the next proof.

Language and Definition
Clear and precise language is necessary, not just in mathematics, but in every walk of life. Every field, from accounting to education to zoology, has its own specialized, technical vocabulary that allows for concise communication. Mathematics is no different. However, in mathematics, definition is also a skill.

Mathematicians use logic to determine what to include in a complete definition of a geometric object or relationship, and distinguish from that list of facts any results that can be deduced from that list. Rectangles, for example, have many properties, but which of those properties would constitute a complete definition? Is there only one such list?

Students will not only learn and use definitions, but will analyze and invent them. They will expand their understanding of definitions by thinking of extreme cases. For example, a trapezoid in which one of the
bases is very small resembles a triangle, but a trapezoid in which the bases are very close in length resembles a parallelogram. Both of these insights lead to a more complete understanding of trapezoids.

Students will also learn to craft logical arguments, both informally and in more formal proofs. Such arguments depend on precise language as well as appropriate choices about what information to include, paying attention to the background expected from the audience. Attention to language and definition is a critical part of mathematical proof.

**Studying Completed Proofs**

Another way that students develop their ability to write mathematical proofs is through studying proofs completed by others and working to understand the arguments thoroughly. Students find that this is a very active process. It usually involves stopping to ask questions, and working through examples. Also, students should make sure that each step in the proof makes sense before going on to the next.

Often, students study different ways of proving the same result. An example is Lesson 3.11, Pick a Proof, in which students study three different dissection proofs of the Pythagorean Theorem.

A variation of this occurs when new concepts and methods, such as vectors in the coordinate plane, provide opportunities for new proofs. Students use these new ideas to find another pathway to the proof of an important result, such as the concurrence of medians in a triangle. The new technique yields different insights.

Students who prove results in more than one way gain a feel for the varied approaches that may be possible in a mathematical situation. Multiple proofs let students play to their strengths (for example, by contrasting arguments that are more visual or more computational). They let students find the way that makes the most sense for them. Students’ new understandings of a theorem will help them follow additional arguments that are less natural for them but every bit as legitimate.

Students might also study a completed proof as part of a chapter project. A series of questions and tasks guide students through a proof of a result that is a bit beyond their reach to prove independently. This process can introduce students to some special techniques, such as computing area in two different ways, which may come in handy in a future proof.

**Visualization**

Visualization is an important habit of mind in mathematics. Being able to picture a situation makes it more real—more concrete. CME Project *Geometry* provides many situations that help students develop this skill. For familiar things, students will make, view, and manipulate mental images or models they can touch or draw in order to analyze and review their work. For things they have never seen before, they will also interpret, or even develop, images.

There are times when *Geometry* purposely presents problems without diagrams. It is the intent that students learn to extract pertinent information and find their own ways of picturing a situation.

All of these experiences will help advance students’ abilities to visualize spatial and quantitative information and to analyze information that is presented visually.

In Chapter 1, students gain extensive experience in picturing, drawing, constructing, describing, and reasoning about shapes. For example, Lesson 1.0, Habits of Mind, asks students to think about unfolding a paper cube and laying out its faces as a set of squares attached at their edges. A figure that results from unfolding
the cube in this way is a net for the cube.
From Geometry page 6:

6. Which of the following are nets for cubes. How do you know?

   a. 
   
   b. 
   
   c. 
   
   d. 

7. How many different nets for a cube can you make? What does it mean for nets to be different?

Geometry also asks students to analyze and interpret diagrams that communicate information about quantitative, spatial, and other relationships. It asks students to interpret drawings that depict proofs without words, teasing out the mathematical assumptions that underlie the pictures. The following dissection “proof without words” of the Pythagorean Theorem is from Geometry page 224.
Proof in a Series of Specialized Domains

The sequence of topics in CME Project Geometry is strategic, because it must present theorems in a provable order. In essence, theorems have prerequisites—established preliminary results—that are sufficient for proving the theorem. If you use a statement as if it were true before you have proven it, you must consider it (and all that follows from it) as only conjecture.

Chapter 1 gives students informal exposure to many of the topics they will encounter throughout the year. It is different from a more formal beginning. It uses without proof many statements as if they were true. So you need to teach and assess Chapter 1 in its own special way.

In Chapter 1, students experiment and develop their own theories and conjectures. They get an idea of what kinds of questions they will ask and answer in geometry. Students write only informal arguments. They learn about what constitutes a complete and correct argument from your feedback, where you point out open questions or incorrect logic in their work, and from constructively critiquing their classmates’ arguments.

Formal proof begins in Chapter 2, “Congruence and Proof.” At first it is restricted to ideas of triangle congruence. Triangles are objects that students can construct either with geometry software or with compass and straightedge. They can physically compare two triangles, reinforcing the meaning of congruence. Statements relating to triangle congruence constitute a limited set of postulates and theorems with a clear objective.

Triangle congruence is a good place to start with proof, but a key idea is left as a conjecture at first. Students do not immediately have a way to prove that the sum of the measures of the angles in a triangle is 180°.

Then, in Investigation 2B, “Proof and Parallel Lines,” students accept postulates and prove theorems in another arena—the angles formed when parallel lines are cut by a transversal. The Parallel Postulate allows them to prove the Triangle Angle-Sum Theorem among other results.

In Chapter 3, “Dissections and Area,” students encounter a new idea. Rather than proving something about a static diagram, they prove something about a process. Students develop their own algorithms for cutting a polygon of one type into pieces and rearranging those pieces to form a different type of polygon.

They need to clearly describe the properties of each cut they make and then verify that the pieces fit exactly as required. Area is unchanged (invariant) in a dissection like this, so these dissection algorithms form the basis for the area formulas for a variety of geometric shapes.

Chapter 4, “Similarity,” relies on everything that has come before. Students understand similarity in relation to congruence. In fact, students were probably ready to talk about similar triangles as soon as they found that AAA does not work as a triangle congruence test. However, they would have been unable to prove that similar triangles have a constant ratio between the lengths of corresponding sides without theorems about parallel lines and area formulas.

Area is critical to the proof of the Parallel Side-Splitter Theorem. This, in turn, leads to proofs of the tests for similarity in triangles that rely on the fact that the ratio between corresponding side lengths in similar figures is constant.

Chapter 5, “Circles,” includes theorems about circles and about lines and angles intersecting circles.

Since this course does not formally cover limits, there are some places where students must use informal reasoning by continuity to develop formulas for area and circumference. For example, students consider limits for area and perimeter of regular polygons—inscribed in and circumscribed around circles—as the number of sides increases.
A unique feature of this chapter is its definition of $\pi$—the area of a circle of radius 1.

This definition allows students to capitalize on their work with dilations to make the area formula for circles a basic and obvious result. Students already know that when an object of area $A$ is scaled by a factor $k$, the area of the scaled copy is $k^2A$. Scale a circle of radius 1 by a factor $r$ and the image is a circle of radius $r$. If the area of the original circle is $\pi$, the area of the scaled copy must be $\pi r^2$.

Students who are familiar with other, perhaps more traditional, definitions of $\pi$ can show that the definitions are equivalent.

In Chapter 6, “Using Similarity,” students continue to build on their earlier work to prove many classical geometric theorems. They also begin to develop trigonometry by defining the trigonometric functions as ratios of side lengths in right triangles.

Chapter 7, “Coordinates and Vectors,” connects geometry to students’ earlier experiences in algebra through the Cartesian coordinate plane.

In Chapter 8, “Optimization,” students connect and relate many of the ideas they have explored over the entire course in a study of geometric optimization.

**Invariants**

A question that is prevalent throughout mathematics is, “Does this always happen?”

Students’ first conjectures often center on questions like this. Their initial notion of what mathematical research is may be, in effect, a search for invariants. And once they have proven that a quantity, such as the power of a point, is invariant, they can use that to develop further results.

They can also develop a process in such a way that it holds some quantity invariant.

**Experimentation and Conjecture**

Geometry software provides a very rich environment for students to experiment and tinker with geometric objects. It is easy to hold some points, shapes, or properties constant while others vary. Students can show measurements of lengths, angles, and areas on the screen and see how these measurements vary as objects move and shapes change.

(If students do not have access to geometry software, you can get some of the effect by having each student produce a general construction. Then compare these constructions across the classroom.)

In all of this motion, an invariant is striking. Students are likely to notice and wonder about something that seems to remain constant over many different constructions or calculations. They will ask the question, “Does this always happen?”

You have to recognize these moments and help students identify these questions as conjectures.

Conjecturing is an intrinsically mathematical activity. It is a research mathematician’s task to come up with a conjecture (or sometimes, several conjectures) to investigate within a particular mathematical situation.

Perhaps more important, when students are the source of a question, they are more personally invested in finding the answer. Owning the question motivates the search for a proof.

**Invariance as a Defining Quality**

CME Project Algebra 1 defines a line as a set of points for which the slope between any two of the points is invariant. In Geometry, students use similar triangles to prove that this invariant slope guarantees collinearity.

Students also encounter the numerical invariant that is the power of a point with respect to a circle. Suppose you are given a
circle $C$, a point $P$, and a chord through $P$. Then $P$ splits the chord into two segments. The product of the lengths of the two pieces is the power of $P$, or $P(P)$. $P(P)$ is invariant for all chords through $P$ in this circle.

Once students have identified an invariant like this, they can capitalize on the fact that it remains constant to prove further results.

**Holding Area Constant**

Each dissection that students develop in Chapter 3 keeps the area of the figure the same at each step in the process. If you start with a shape having a known area, then you also know the area of each of its potential dissections.

If, instead, you start with a shape of unknown area but with known dimensions, you may be able to dissect to get a simpler figure with known dimensions. If you can compute its area, you will find the area of the original shape.

This is how students prove area formulas for parallelograms, triangles, and trapezoids, and also how they prove some other theorems, including the Pythagorean Theorem.

**Dilation**

In CME Project Geometry, two figures are defined to be similar if one is congruent to a dilation of the other. This definition is equivalent to the one you likely learned as a student: Two figures as similar if all pairs of corresponding angles are congruent and all pairs of corresponding sides are proportional.

The reason CME Project Geometry uses the dilation definition is to give students practical experience with pairs of similar figures before formally defining the relationship.

Students first experience similarity in the context of scale drawings. They learn to construct scaled copies through two dilation methods—the ratio method and the parallel method. By doing these dilations, students get a feel for what similarity is and how lengths and angle measures in a pair of scaled copies are related.

Students then prove that the two methods used to produce dilations give the same result by proving the Side-Splitter Theorems. To prove these theorems, students look closely at nested triangles and other nested shapes. Students see that if they place one figure inside one of its dilations and line up some corresponding sides, other pairs of corresponding sides are parallel. This is the type of configuration of similar figures that is often most helpful in further proofs.

After building this spatial and numerical understanding of similarity through considerable experience with dilations, Geometry formally defines similarity in the fourth, rather than the first, investigation of Chapter 4.

**Proving Formulas**

An overriding goal of the CME Project is that students come to expect mathematics to make sense. Nothing should drop out of the blue. Each new fact or process should be developed from students’ prior knowledge. As a consequence, formulas should always be justified based on experience and, where possible, on formal proof.

**Dissection Algorithms**

By the time students reach high school geometry, they likely have learned and used area formulas for many different kinds of figures. However, there are often basic misconceptions about these formulas, such as what is meant by the height of a triangle. Students may not have a strong understanding of why a formula works or what it is really measuring.

CME Project Geometry approaches area formulas with a series of puzzles that rely on students’ recently refreshed knowledge.
about congruence and parallel lines. *Geometry* asks students to find ways to cut a polygon and rearrange the pieces to form a different kind of polygon. Students proceed from informal strategies into more formal algorithms that carefully describe where to cut and how to move the pieces, and they prove that the pieces fit together exactly. Each new figure must have no gaps or overlaps between the pieces.

Students also have to prove that lines that appear to be straight in a rearrangement actually are straight, and that sides that appear to fit together actually have the same length.

Students then can conclude that the area of the figure before the cuts and the area of the figure formed by the rearrangement are the same.

Here is an example of how dissections lead to the formula for the area of a trapezoid.

You cut the trapezoid along the segment joining the midpoints of its nonparallel sides. Then you cut along a perpendicular from one of the vertices of the smaller base to the first cut. Then rearrange the pieces.

Students prove that

- the pieces fit as shown
- the midline cut is parallel to the bases of the trapezoid
- the height of the rectangle formed is half the height of the original trapezoid
- the base of the rectangle is equal to the sum of the bases of the trapezoid

Thus the area of a trapezoid with bases $b_1$ and $b_2$ and height $h$ is the same as the area of a rectangle with base $b_1 + b_2$ and height $\frac{1}{2}h$, or $\frac{1}{2}h(b_1 + b_2)$.

Notice that the dissection of the trapezoid relies on the possibility of cutting along a perpendicular from a vertex to the midline of the trapezoid. A complete derivation for this formula would also include strategies for dissecting trapezoids where such a cut is not possible.

You can get around this problem by dissecting the trapezoid directly into a parallelogram. Use the midline cut and rotate a piece.

By this time, students have already proven the area formula for a parallelogram by dissecting it into a rectangle using a cut along an internal altitude. Unlike trapezoids, parallelograms always have an internal altitude on their longest side.

Now this does not mean that the area of a parallelogram is only shown to be equal to the product of the long base and the corresponding height. Students also show that the area of a parallelogram is equal to the product of the short base and its corresponding height, but that part of the proof requires them to deal with the case where there is no internal altitude on the short base.

In this case, students cut the parallelogram along a segment with the midpoints of the
two longer sides as endpoints. Then they put the two pieces together to form a new parallelogram with a base that is twice as great and a height that is half as great.

If the original parallelogram is so tall and narrow that this dissection still does not produce a new parallelogram with an internal altitude, students can repeat the process as many times as they need to.

**Cavalieri's Principle**

Volume formulas are a bit trickier to derive because three-dimensional dissections are much more difficult for students to visualize and to model than two-dimensional ones.

Instead, in this course, students use Cavalieri’s Principle to see solids in a different way. The technique has an advantage in that it prepares students for the ways they will think about volume in a calculus course.

**Theorem 6.3 Cavalieri's Principle**

Two solids of the same height are cut by a plane so that the resulting cross sections have the same area. If the solids also have cross-sectional areas equal to each other when cut by any plane parallel to the first, then they have the same volume.

This means that students think of a solid as a stack of thin sheets—cross sections.

They see that the volume of any prism is equal to the area of its base multiplied by its height. This generalizes beyond prisms to include any solid with a constant cross section and this cross section can have any shape at all, even a strange or curved shape.

Similarly, the volumes of all pyramids of equal height with bases of equal area have the same volume. The trick here is finding the volume of any one pyramid. Students see how to dissect a cube into three congruent square pyramids with height equal to the length of the base.

This allows them to conclude that a square pyramid with height equal to the side length of its base has volume equal to one-third the volume of the cube with the same base.
They generalize this to calculate the volume of a rectangular pyramid of any height using a scaling argument.

Finally, as with the prism, this volume formula generalizes to pyramid-like solids with triangular vertical cross-sections, which can have any shape for a base.

The capstone of the work with Cavalieri’s Principle is its use to derive the formula for the volume of a sphere. Students relate the circular cross sections of the sphere to the ring-shaped cross sections of a cylinder with a double cone cut out of it.

To do this, students use triangles in the vertical cross sections of the solids to relate to $h$ both the radius of the circular cross section of the sphere and the inner radius of the ring cross section of the cylinder and cone.

Then they show that at every height $h$ the areas of the cross sections of the solids are equal.

**Connections to Algebra**

In Chapters 7 and 8 of CME Project Geometry, students bring together ideas from algebra 1 and geometry to make a start on studying analytic geometry. They will continue this work in algebra 2 and precalculus, as they begin to regard the graphs of functions as geometric objects that they can transform in the same ways that they transform geometric objects, and that there are algebraic ways to express these geometric transformations.

**Cartesian Coordinates**

The Cartesian coordinate system is a basic way to connect geometry and algebra. By identifying each point in the plane with an ordered pair of numbers based on a coordinate grid, a geometric object now has location and orientation with respect to the coordinate axes, as well as shape and size. This allows students to use geometry to prove many of the results they used when graphing in algebra 1.
For example, from the endpoints of a given general segment in the plane, students construct lines that are parallel to the axes of the coordinate system. These lines form a right triangle with the given segment as hypotenuse. By using the Pythagorean Theorem and the general coordinates of the endpoints, students derive formulas for finding the length of such a segment and the coordinates of its midpoint.

The given segment also has slope relative to the coordinate axes. Students learn to measure this slope both as a ratio of vertical change to horizontal change and (through their initial study of trigonometry) as the tangent of an angle.

Students also use similar triangles to prove that three points are collinear if and only if the slopes between two pairs of them are equal, and to prove that the product of the slopes of two perpendicular lines is $-1$.

**Vector Proofs**

Students gain insight into the geometry theorems that they prove by proving them in a new way—in the coordinate plane, through the use of vectors.

Students first encounter vectors as a way to describe translations in the coordinate plane. They think of a translation as a mapping, such as $(x, y) \rightarrow (x + 7, y + 2)$, that adds the coordinates $(7, 2)$ to the coordinates of each point in the original object. The effect of this mapping is to make a congruent image of the object that is 7 units right and 2 units up from the original.

Students draw arrows from points in the original object to their corresponding points in the translated image. All of these arrows are congruent and point in the same direction. Later students will see this collection of arrows as one way to think of the vector $(7, 2)$. The translation is another.

CME Project *Geometry* formally introduces vectors as arrows in the plane, or line segments with direction. Over time, *Geometry* expands this understanding so that all arrows that have the same length and slope, or orientation, are equivalent. Students learn to write the equation of a line in vector form, by using any point on the line and any vector with the same slope as the line.

This vector form has some advantages that make it ideal for proving certain geometric results. With the vector form of a line, it is a simple matter to cut a segment $AB$ at a point $P$ so that $P$ is “a $k$th-of-the-way” from $A$ to $B$. This means that $$\frac{AP}{AB} = k$$

Because of this, in the vector proof of the concurrence of medians in a triangle, students are able to show not only that the medians all intersect in a point, but that the intersection point is two-thirds of the way along each median from each vertex to the midpoint of the opposite side.
Students can also use the vector equation of a line in three (or more) dimensions, whereas the standard equations for lines in two dimensions do not extend to the three-dimensional environment.

The Development of Trigonometry

Every triangle has six associated measurements—the lengths of its sides and the measures of its angles.

Students have learned that they can often completely determine a triangle with only three of these measurements. The triangle congruence tests tell which three pieces of information will be enough.

Students have used these tests over and over to prove theorems. But there is something more going on.

If you can completely determine a triangle with two side lengths and their included angle, SAS for example, then there should be a way to figure out what the other side length of the triangle is and what the measures of the other two angles are.

The goal of finding those unknown measurements drove the initial study of trigonometry.

CME Project Geometry defines trigonometry as the ratio of a pair of side lengths in a right triangle. As a consequence, the trigonometric ratios are defined only for angles that can occur in a right triangle—angles with measures between 0° and 90°.

But that restriction does not hold back students in their quests to find unknown measurements in nonright triangles. They can use an altitude of the triangle to form right triangles that share sides and angles with the original triangle.

Eventually, given enough information to completely determine a triangle, students are able to find all of its measurements—they have solved the triangle. That is the end of the beginning of their work with trigonometry in the CME Project.

Optimization

Chapter 8, “Optimization” focuses on optimization problems—finding maximums and minimums. It uses no calculus. It requires some algebraic calculations, but students can solve most problems in the chapter with essentially geometric techniques.

Connections to Calculus

Optimization problems tackle some more of the fundamental questions that students naturally ask about mathematical situations. “Which one is the largest? What is the smallest that it can be? How close can you get?”

Students find that they can apply geometric methods to some problems of this type—traditionally reserved for calculus—sometimes in very elegant ways.

Geometric solutions can be as simple as, “The shortest path between two points is a straight path,” or as subtle as, “If a continuous graph goes up and then comes down, it must have had a maximum value somewhere in between.”

The problems in this chapter help students get at the big ideas behind optimization problems—thinking about extreme cases and boundary conditions and using a mixture of deduction, experimentation, and reasoning by continuity.

Reasoning by Continuity

In fact, the main reason for including this chapter in CME Project Geometry is that it gives students ample opportunity to explore the important geometric habit of mind of reasoning by continuity.

Here are some examples of what it means to reason by continuity.

Is there a line that cuts the area of this blob exactly in half?
Let the area of the blob be 1.

Here is a line to the left of the blob. All of the blob (along with its area) is to the right of this line.

Visualize a series of lines parallel to this one, moving farther and farther to the right until all of the blob is to the left of one of these parallel lines.

Somewhere between the first line and the last line there must be a line that cuts the area of the blob exactly in half—half of the area is to the right and half is to the left. That is because the amount of area to the right of the line changes continuously and goes from 1 to 0. By the continuous nature of the situation, you know that for some line the area on each side of the line must be \( \frac{1}{2} \).

Another problem:

Was there a time in your life when your weight in pounds was equal to your height in inches?

Finding the Shortest Path is an optimization problem in Lesson 8.2.

Suppose you are motorboating on a river and you need fuel. First you must drop a passenger off on one riverbank. Then you must refuel at a station on the other riverbank. Where should you drop off your passenger in order to get to the fuel station by covering the least possible distance?

Students can quickly see that they do not want to drop the passenger off upstream from their current position at \( A \) or downstream from the fueling station at \( B \). And it is not immediately obvious how to find the best drop-off point in between.

By trying several points, students can observe that as \( P \) moves downstream, the distances get shorter for a while and then start getting longer. Reasoning by continuity, they should conclude that the best point is somewhere in the last interval showing decreasing distance or the first interval showing increasing distance.

(A straight line from the reflection image of the motorboat to the fueling station crosses the bank exactly at the best drop-off point.)
Here is an optimization problem (from Lesson 8.12) in multiple-choice format.

Given an equilateral triangle with side length 10 units, find $DR + DQ + DP$. This is the sum of the distances from a point $D$ inside the triangle to the sides of the triangle.

A. 10 units  
B. $5\sqrt{3}$ units  
C. 30 units  
D. 8.6 units

In trying to solve this multiple-choice problem, a clever student named Rich reasoned that since no information is given about $D$'s location, and since one of the answers must be correct, the "sum of the distances to the sides" function from any point inside the equilateral triangle must be constant in the triangle's interior.

Rich reasoned that he could choose $D$ to be anywhere he wanted. Rich also assumed that the sum of the distances to the sides of the triangle varies continuously.

So, moving $D$ very close to a vertex and thinking continuity, Rich reasoned that the sum of the distances from $D$ to the sides must be the same as the sum of the distances when $D$ is anywhere on a side, including at a vertex.

But at a vertex, two of the distances are zero. The third is the triangle's height. This means that the sum of the distances from a point $D$ inside an equilateral triangle to the sides of the triangle must be equal to the length of an altitude of the triangle, or $5\sqrt{3}$ units.

The Airport Problem is an optimization problem in the chapter's Project.

Three neighboring cities, all about the same size, decide to share the cost of building a new airport. They hire your group as consultants to find a location for the airport.

Let $A$, $B$, and $C$ be the locations of the cities. Let $D$ be the location of the airport so that $DA + DB + DC$ is the least sum possible.

For this problem, students can build a physical model of the situation with a metal ring and string threaded through nails.

They can also model the problem with geometry software.

Through experimentation with either model, they develop the conjecture that the airport should be at a location, if possible, where the three roads from the cities make 120° angles with one another. This point is also known as the Fermat point of a triangle.

Students learn how to construct this point and prove that no other point inside the triangle has a smaller sum of distances to the vertices.

CME Project Algebra 2 and Precalculus pick up on and reinforce this habit of mind as students develop an idea of what it means for a function to be continuous, and what is implied by continuity.
Implementing the CME Project
Mathematical Approaches in Algebra 2

Beginning With Fitting Functions to Tables

Very early in Algebra 2, students see a collection of tables like this one.

<table>
<thead>
<tr>
<th>Input, n</th>
<th>Output, A(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
</tbody>
</table>

A typical algebra 2 student might look at this and think at first, “I don’t remember how to do this! I took algebra 1 ages ago!”

But it can be amazing how much mathematics a student who “doesn’t remember any algebra 1” can generate. It is not that all students look at that table and recognize that \( f(x) = x^2 - 1 \) is a function that fits that table. More often, students will begin with descriptions such as, “the outputs go up by 1, then 3, then 5, then 7, …” As they gain experience, they move on to more formal descriptions of the functions.

CME Project Algebra 1 introduced functions as machines. Although this idea is useful, students in Algebra 2 are to move beyond that metaphor and into thinking of functions as objects in their own right.

Beginning the Algebra 2 course with a set of tables serves three purposes. Students are to

- develop a functional perspective. What is a function?
- become fluent in working with functions; identifying, composing, and (when possible) inverting functions
- develop an algebraic perspective in the sense of thinking of functions as algebraic objects

What does it mean to “think of functions as algebraic objects”?

The set of functions has an operation, composition, for which the following are true.

- The set is closed. The composition of two functions is a function.
- There is an identity element. For any function \( f \), if \( g(x) = x \), then \( f \circ g = g \circ f = f \).
- The set is associative. For any three functions \( f, g, \) and \( h \),
  \[
  f \circ (g \circ h) = (f \circ g) \circ h.
  \]
- Some elements have inverses. For example, \( f(x) = 3x + 5 \) has an inverse function, while \( g(x) = 3 \) does not.

Other properties do not necessarily hold. For example, the set is not commutative, because it is not always true that \( f \circ g = g \circ f \).

CME Project Algebra 2 does not focus on this in a formal way. The work here is more about developing an algebraic perspective that will serve students as they work with other sets in the future. (For example, this will smooth the way for students’ work with matrices in Chapter 4.)

Difference Tables

Algebra 2 spends a fair amount of time looking at difference tables. Many textbooks construct difference tables with the differences hovering between the function values. For example, given a table like this,

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
</tbody>
</table>
an algebra 2 book might show the differences this way. (Here, \( \Delta \) means difference. \( \Delta^2 \) will mean "second difference" or the difference of the differences.)

CME Project Algebra 2 shows difference tables this way:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( A(n) )</th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>( 0 - (-1) = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( 3 - 0 = 3 )</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>( 8 - 3 = 5 )</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>( 15 - 8 = 7 )</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

There are reasons for writing differences between function values \( f(n) \) and \( f(n - 1) \) on the same line as \( f(n - 1) \), including:

- It is easier to think of \( \Delta \) (and \( \Delta^2 \) and \( \Delta^3 \)) as functions in their own right if it is possible to see the output next to the corresponding input.
- This table structure will serve well in precalculus and beyond. For example, in precalculus, students will use Newton’s Difference Formula to fit polynomials to tables. Formatting difference tables in this way will be useful. And the notion of “difference function” is important to calculus.

**Lagrange Interpolation**

In middle school and high school, students get experience finding functions that fit tables having consecutive inputs, and outputs with constant first (or second, or third) differences, or even constant ratios. But how does that procedure work with more complicated tables? Consider, for example, this one.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( k(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
</tr>
</tbody>
</table>

By the end of Algebra 2, students will be able to use systems of equations to find a quadratic function that fits this table. But there is another method, Lagrange interpolation, that allows students to find a polynomial function that fits any table they encounter.

To find a function that fits the table above, find functions that fit the following three tables.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( j(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( n(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( m(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
</tr>
</tbody>
</table>

These tables do not have consecutive inputs, but you can use the zeros to help you find functions that fit them.

- A function of the form

  \[ j(x) = c(x - 3)(x - 6) \]

  fits the first table, since 3 and 6 are both zeros. Use \( j(1) = 20 \) to find \( c = 2 \).

- A function of the form

  \[ n(x) = c(x - 1)(x - 6) \]

  fits the second table, since 1 and 6 are both zeros. Use \( n(3) = 12 \) to find \( c = -2 \).
A function of the form

\[ m(x) = c(x - 1)(x - 3) \]

fits the third table, since 1 and 3 are both zeros. Then use \( m(6) = 60 \) to find \( c = 4 \).

To find a polynomial that fits the original table, simply add:

- Given any table, students can use Lagrange interpolation to find a polynomial that fits the table. The method always works.
- Students review the important Zero Product Property from algebra 1. They preview the Factor Theorem. Working with Lagrange interpolation gives students experience with the relationship between factors and roots of polynomials.
- Students get used to operating with polynomials. They begin to develop intuition about degrees of sums and products of polynomials.
- Students work with one of the most fundamental concepts of algebra: linear combinations. Students use linear combinations of polynomials to create new polynomials that interpolate tables.
- Students begin to characterize all polynomials that interpolate a given table.
- Teachers in many, many field-test sites have said that students love doing this.

One field-test teacher, Chris Martino, has described the impact of Lagrange interpolation in the following way:

I didn't even really remember what Lagrange interpolation was, so I didn't really want to bother . . . But I couldn't believe the connections my students made when they started working on it. I was floored—they made connections that they had never made before. Light bulbs were going off everywhere . . . They understood how to add functions, and why you might want to. They understood that functions are things you can add. They saw some value to factoring, because they really understood the relationship between factors and roots. And what surprised me most of all was how much they loved solving the problems—because they were good at it.

### Advanced Factoring

Many students who can factor monic (leading-coefficient-1) quadratics by the sum-and-product approach often have much more difficulty when the leading coefficient is not 1. One approach here is to use the correspondence between roots and factors and to employ the quadratic formula.

To factor a quadratic polynomial, set it equal to 0. Use the quadratic formula to find the roots. Then reconstruct the factors from the roots. It always works. But it involves a fair amount of computation. And it depends on the existence of a formula for solving the equation—something that does not exist for polynomials of degree higher than 4.

The CME Project wants to develop general-purpose methods that live beyond algebra 1. One such method starts with the observation that you can lump

\[ 4x^2 + 36x + 45 \]

into

\[ (2x)^2 + 18(2x) + 45 \]
You can think of this as “quadratic in 2x,” making 2x the variable. You can even replace 2x with some symbol, say z. (Algebra 2 uses a picture of a hand.)

Write the quadratic as
\[ z^2 + 18z + 45 \]
which you can then easily factor as
\[ (z + 15)(z + 3) \]
Replacing z with 2x gives the factorization of the original quadratic
\[ (2x + 15)(2x + 3) \]
But what can you do if the quadratic is like this one?
\[ 6x^2 + 11x - 10 \]
Students can reason this way.

- Multiply the polynomial by 6 to make the leading coefficient a perfect square. Remember that you have to divide by 6 at some point to get back to where you started.
\[ 6(6x^2 + 11x - 10) = (6x)^2 + 11(6x) - 60 \]

- This is a quadratic in 6x. Set \( z = 6x \) to produce a monic quadratic:
\[ z^2 + 11z - 60 \]

- This factors.
\[ (z + 15)(z - 4) \]

- But \( z = 6x \), so you have
\[ (6x + 15)(6x - 4) \]

- Pull out common factors, 3 from the first binomial and 2 from the second.
\[ 6(2x + 5)(3x - 2) \]

- Dividing by 6 gives the factorization of the original polynomial
\[ (2x + 5)(3x - 2) \]

This is the scaling method for factoring nonmonic polynomials. It has the benefit of reducing a complicated problem to one that is simpler to solve. You can extend this scaling method to polynomials of any degree.

In general, if
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \]
then
\[ a^{n-1}_n f(x) = (a_n x)^n + a_{n-1} (a_n x)^{n-1} + \ldots + a^2_1 (a_n x) + a_0 \]
Letting \( z = a_n x \), you have a monic polynomial in \( z \).

## Algebra With Functions, Polynomials, Matrices, and Complex Numbers

Over several centuries, algebra evolved from the study of methods for solving numerical equations—first in the positive integers and later in the real numbers—to a search for general methods (like the quadratic formula) that apply to whole classes of numerical equations. Along the way, mathematicians and scientists began looking at equations with solutions that lie in other kinds of systems, equations with solutions that are matrices, functions, polynomials, permutations, complex numbers, and more general kinds of “numbers.”

This led researchers to investigate the extent to which the rules for calculation with numbers—the Basic Rules of Arithmetic in CME Project Algebra 1—hold in other systems. They saw that while many rules and their consequences extend to other systems that come up in nature, many do not.

In the real numbers, for example, squares are never negative, a fact that is useful over and over when analyzing maximums and minimums for quadratic functions. But in the complex numbers, \( i^2 = -1 \).

There are even useful systems in which the Zero Product Property (if the product of two things is zero, one must be zero) fails. For example, in the system of 2-by-2 matrices with real entries,
\[
\begin{pmatrix}
2 & -2 \\
2 & -2
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
1 & -1
\end{pmatrix} =
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]
So two nonzero matrices can multiply to zero.
The main application of the ZPP in *Algebra 1* is to show that every quadratic equation has at most two real roots, and that you can find the roots by factoring the quadratic. So, if

\[ x^2 - 1 = 0 \]

then

\[ (x - 1)(x + 1) = 0. \]

By the ZPP,

\[ x - 1 = 0 \text{ or } x + 1 = 0 \]

so,

\[ x = 1 \text{ or } x = -1. \]

In systems without the ZPP, quadratic equations can have more than two roots. For example, the matrix equation analog of

\[ x^2 - 1 = 0 \]

is

\[ A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

And this has at least three solutions,

\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]

In fact, there are many more than three. (Can you find some others?)

This line of reasoning led algebraists to develop a “systems approach” to calculation, leading to the current emphasis in modern algebra on algebraic structure—the study of the rules for calculation in systems that have one or more “binary operations” such as addition and multiplication.

This theme of algebraic structure is in the background of much of the CME Project approach to algebra in high school, although it comes through in the student text in a very informal way. The theme of extension discussed elsewhere in this guide is a manifestation of this approach.

- What are the structural similarities between the ordinary integers and the system of polynomials in one variable that allow you to factor into primes and perform long division?

A good example of the systems approach is the way CME Project *Algebra 2* introduces complex numbers. If you watch beginning students calculate with complex numbers, they act as if \( i \) is like \( x \)—they calculate with the expressions as if they were polynomials, replacing \( i^2 \) with \(-1\) along the way or at the end.

In fact, they are noticing (as did 16th-century mathematicians) that the algebraic structure of the complex numbers resembles the algebraic structure of polynomials, with an extra simplification rule. This is the motivation for the CME Project definition of complex numbers in Chapter 3 of *Algebra 2* and the approach to roots of unity later in *Precalculus*.

Regarding polynomials themselves, students in algebra 1 learn the Form Implies Function Principle: If two polynomials are equivalent under the basic rules, they define the same function. The converse is not true in general. There are algebraic systems in which two polynomials define the same function, but the polynomials cannot be transformed into each other with the basic rules.

But there is a partial converse for polynomials that is true. *Algebra 2* shows that if two polynomial functions agree for enough inputs, they are in fact equivalent under the basic rules. More precisely, if two polynomials of degree \( n \) agree for \( n + 1 \) inputs, they are identically equal (and hence define the same function). This is quite a wonderful fact, one that finds uses in many parts of mathematics.

For example, if two cubic polynomials agree at four inputs, they agree at every real number—a Function Implies Form Principle for polynomials.
Graphs and Transformations

Starting with Algebra 1, the CME Project texts take the stance that students can start with a set of core graphs and learn to transform them in consistent ways. The graphs of \( y = (x + 3)^2 \) and \( y = x^2 \) are related in exactly the same way as the graphs of \( (x + 3)^2 + y^2 = 25 \) and \( x^2 + y^2 = 25 \).

The CME Project bases development of graphs on two components.

- A set of core graphs: In Algebra 1, this core set includes the graphs of
  
  \[
  y = cx \\
  xy = c \\
  y = x^2 \\
  y = x^3 \\
  y = \sqrt{x} \\
  y = |x|
  \]

  In Algebra 2, this list expands to include
  
  \[
  y = b^x \\
  y = \log_b x \\
  x^2 + y^2 = 1 \\
  y = x^3 \pm x \\
  y = \sin x \\
  y = \cos x
  \]

  In Precalculus, this list expands again to include general polynomial and rational functions, other trigonometric functions, and other conic sections.

- A set of transformations: This set includes translations, reflections, and scaling.

What makes this treatment different from traditional treatments?

- CME Project Algebra 1 introduces the notion of transforming graphs before attaching those transformations to any particular type of graph. Students get used to transforming basic graphs before distinguishing between, for example, transforming circles and transforming parabolas.

- CME Project Algebra 2 develops a second way of looking at transformations. Usually students think about moving graphs, but in Algebra 2, students see as well that they might simply relabel axes. They get practice doing this for any type of transformation.

Students see a hint of this in Algebra 1, but delve into it in earnest in Algebra 2. This focus on the axes addresses an issue students sometimes face. They think, “Everything is backwards in transformations! Why is it that when I move something three units to the right, I write \( (x - 3) \)? Shouldn’t I be writing \( (x + 3) \)?”

CME Project provides a second interpretation. You are not moving the graph three units to the right, you are moving the axis three units to the left.

Ultimately, this relabeling of the axes generalizes to the important idea of changing the coordinate system, which is central to linear algebra and multivariable calculus. It also allows students to reduce every cubic equation to one of three core forms.

Why Affine Transformations?

An affine transformation is a function of the form \( x \mapsto ax + b \) where \( a \) and \( b \) are real numbers with \( a \neq 0 \).

For example, the function

\[
\begin{align*}
  x &\mapsto 2x - 5
\end{align*}
\]

(which the text denotes as \( A_{2,-5} \)) is an example of an affine transformation.

CME Project (in Chapter 6) views these affine transformations not just as functions, but as mathematical objects. What is more, the set

\[
G = \{ A_{a,b} : a, b \in \mathbb{R}, a \neq 0 \}
\]

of these transformations comes with an algebraic structure. Just as you can combine elements of the set of integers (by addition, say), you can combine the elements of \( G \) by function composition.
Here is an example from Chapter 6.

For a real number $x$, we have

$$(A_{2, 3} \circ A_{-1, 4})(x) = A_{2, 3}(A_{-1, 4}(x))$$

$$= A_{2, 3}(-x + 4)$$

$$= 2(-x + 4) + 3$$

$$= -2x + 11$$

$$= A_{-2, 11}(x)$$

Therefore $A_{2, 3} \circ A_{-1, 4} = A_{-2, 11}$.

The set $G$ here is actually an example of an algebraic structure known as a group. The study of groups is an essential part of higher mathematics. This chapter gives students the opportunity to explore this concept in a concrete setting.

Here are some ideas from group theory that they will encounter:

- The set $G$ is closed. That is, if you combine two affine transformations, you get another affine transformation.
- The set $G$ has an identity element, namely $A_{1, 0}$ (that is, the function $x \mapsto x$). The identity has the following special property:

$$A_{1, 0} \circ A_{a, b} = A_{a, b} \circ A_{1, 0} = A_{a, b}$$

- Every element of $G$ has an inverse—the inverse of $A_{a, b}$ undoes the effect of $A_{a, b}$.
- The elements of $G$ do not commute. In high school algebra, students often talk about the commutative property but rarely encounter examples of algebraic systems that are noncommutative. We have seen that

$$A_{2, 3} \circ A_{-1, 4} = A_{-2, 11} \neq A_{-1, 4} \circ A_{2, 3}$$

It turns out that $A_{-1, 4} \circ A_{2, 3} = A_{-2, 11}$.

Therefore,

$$A_{2, 3} \circ A_{-1, 4} \neq A_{-1, 4} \circ A_{2, 3}.$$  

Another important idea that CME Project Algebra 2 explores is the comparison of algebraic systems.

For any real number $a$, let $T_a$ denote the function $x \mapsto x + a$. We call $T_a$ a translation by $a$. First observe that translations are just special types of affine transformations, because $T_a = A_{1, a}$. But there is more.

Let $T$ denote the set of all translations. Then $T$ is more than just a subset of $G$. It comes with its own algebraic structure.

The set $T$ is closed and has an identity element. Also, every element of $T$ has an inverse. In other words, $T$ is a subgroup of $G$. Moreover, consider the following Minds in Action dialog from Algebra 2 Chapter 6.

Tony and Sasha are talking during lunch about what they have learned so far.

Tony So, for translations, we have some properties that seem very familiar.

- $T_a \circ T_b = T_b \circ T_a = T_{a+b}$
- $T_0 = id$
- $(T_a)^{-1} = T_{-a}$

Sasha These do seem familiar. They seem like the properties of addition with real numbers. And look at the properties of dilations.

- $D_s \circ D_t = D_t \circ D_s = D_{st}$
- $D_1 = id$
- $(D_s)^{-1} = D_{s^{-1}}$

They seem a lot like multiplication.

In this dialog, Tony and Sasha are attending to algebraic structure. The properties that Tony identifies in the set of translations indeed resemble the properties of the set of real numbers under addition, as Sasha points out.

And in fact, the two groups—$T$ (under function composition) and $\mathbb{R}$ (under addition)—are algebraically equivalent. In group theory, $T$ and $\mathbb{R}$ are said to be isomorphic.

Similarly, the set $D$ consisting of dilations $D_s$ (i.e., functions of the form $x \mapsto sx$ with $s \neq 0$) is a subgroup of $G$ which is isomorphic to the group of nonzero real numbers under multiplication.
Applications of Affine Transformations

Transforming Equations
In CME Project Algebra 1, students learn about completing the square and the scaling method for analyzing quadratics. These are examples of transforming equations. That is, they transform an equation of the form \( y = ax^2 + bx + c \) into one with the form \( y = x^2 \).

CME Project Algebra 2 revisits this idea in Chapter 6 with the goal of applying it to general cubics. Using dilations and translations, you can transform every equation of the form
\[
y = ax^3 + bx^2 + cx + d
\]
into an equation having one of the core cubic forms
\[
y = x^3, \quad y = x^3 + x, \quad \text{or} \quad y = x^3 - x.
\]
Students then can graph the core cubic using axes that have been similarly transformed to get a graph of the original equation.

Here is an example from the chapter.

Start with the equation
\[
y = 10x^3 - 9x^2 - 13x + 6
\]
Make these substitutions.

- \( M = (D_2 \circ T_{-3} \circ D_{10})(x) \)
- \( N = (D_2 \circ T_{-156} \circ D_{100})(y) \)

with \( \alpha = \frac{1}{\sqrt{157}} \).

You obtain the transformed equation
\[
N = M^3 - M.
\]
The second purpose is to prepare students for a later exploration of how the changes of variables from \( x \) and \( y \) to \( M \) and \( N \) relate the graphs of
\[
y = 10x^3 - 9x^2 - 13x + 6 \quad \text{and} \quad N = M^3 - M.
\]

Replacing-the-Axes Method
As mentioned previously, one component on which the CME Project bases development of graphs is a set of transformations that includes translations, reflections, and scaling. Here is an example from Chapter 6 of how students learn to use affine transformations to translate, scale, and reflect.

Suppose \( M = 2x - 3 \), that is, \( M = A_{(2,-3)}(x) \). Thus, you get
\[
x = (A_{(2,-3)})^{-1}(M)
\]
\[
= (A_{\frac{1}{2}} \circ D_2)(M)
\]
\[
= (T_{\frac{2}{3}} \circ D_2)(M)
\]
Students learn that the transformation \( T_{\frac{2}{3}} \circ D_2 \) has the effect of stretching the number line for \( M \) by a factor of 2 and then shifting it to the left by \( \frac{3}{2} \) units, as shown here.

You align the number lines for \( M \) and \( x \) so that they reflect the relationship between \( M \) and \( x \). Now suppose you want to graph the equation
\[
y = (2x - 3)^2
\]
If we set \( M = A_{(2,-3)}(x) \), the equation becomes
\[
y = M^2,
\]
the core parabola. So in order to graph \( y = (2x - 3)^2 \), you can first graph \( y = M^2 \) and then simply replace the \( M \)-axis with the \( x \)-axis.

**Why Give Complex Numbers \( \frac{1}{8} \) of the Book?**

Complex numbers are numbers of the form \( a + bi \), where \( a \) and \( b \) are real numbers and \( i^2 = -1 \). Most texts introduce the symbol \( i \) as a way to solve quadratic equations with no real roots. The first example is invariably

\[ x^2 + 1 = 0 \]

This masks the history a bit. In fact, the imaginary unit \( i \) and complex numbers first came about in solving cubic equations.

Mathematicians in the 16th century developed a formula for cubic equations similar to the quadratic formula. When you apply this formula to cubic equations with real roots, square roots of negative numbers enter the calculations, only to drop out in the end.

This historical message is an important one. You can invent new mathematical systems for the purpose of solving problems in an existing system. This process of moving up and then back down shows up all over mathematics.

The mathematics of the cubic formula is quite technical for CME Project Algebra 2, but Algebra 2 develops complex numbers in a way that conveys the spirit of the history without the technical details.

The connection between complex numbers and coordinate geometry via the correspondence \( a + bi \leftrightarrow (a, b) \) is one of the richest in mathematics. Students study many of its consequences in Algebra 2 and Precalculus.

**Example: The Geometry of Complex Number Multiplication**

Two wonderful results of this numeric-geometric correspondence are the geometric pictures of addition and multiplication of complex numbers. Addition connects to vector addition. The multiplication rule is more complicated.

Experiments suggest that if \( z \) and \( w \) are complex numbers, the length of \( zw \) is the product of the lengths of \( z \) and \( w \) and the argument of \( zw \) is the sum of the arguments of \( z \) and \( w \). And, in fact, both are true. But getting there usually involves developing and using the addition formulas for the sine and cosine functions.

In the summer of 2002, a group of teachers at the Park City Mathematics Institute in Utah discovered a very simple and elegant way to see what is going on, using nothing more than similar triangles. CME Project Algebra 2 and Precalculus develop that method. Then Precalculus applies it to trigonometric identities.

**Why Teach Complex Numbers Before Trigonometry?**

Studying the complex plane first helps to reduce the level of unfamiliarity of trigonometry. For many students, trigonometry feels completely different from anything they have seen in mathematics. So when they encounter trigonometry for the first time, their work with the complex plane will allow them to make a smoother transition to the topic.

In studying complex numbers, students refresh their knowledge of right-triangle geometry, and develop familiarity with angles from \( 0^\circ \) to \( 360^\circ \). Students work with complex numbers in all four quadrants, so they become familiar with positives and negatives in the different quadrants.
Students have to find, for example, the arguments of $4 + 3i$ and $-4 + 3i$. They come to understand related angles this way.

Conversely, given two complex numbers with the same magnitude and one with, say, argument $40^\circ$ and the other $140^\circ$, their $x$-coordinates will be opposites, but their $y$-coordinates will be the same. Students can then find angles in the other three quadrants to match a reference angle in the first quadrant.

The work with complex numbers also simplifies understanding the unit circle representation of trigonometric concepts. For example, students understand quickly why in Quadrant IV cosine is positive and sine is negative. This also helps students conjecture and prove identities such as

$$\sin (180^\circ - x) = \sin x,$$

which is a consequence of what they have already observed in the complex plane.

Another advantage of this organization is that it lets you reverse the usual sequence and save a great deal of time and overhead in class. Because of the Park City method for deriving the multiplication rule for complex numbers (multiply the lengths and add the arguments), you can use this rule to derive the addition formulas for sine and cosine, rather than the other way around.

In general, being able to use complex numbers in the proof of a trigonometric identity can make the process of establishing such an identity much more efficient.

**Statistics in Algebra 2**

CME Project Algebra 2 continues the theme of connecting statistical ideas with the rest of the curriculum. One of the objectives in the course is to show how to calculate a regression line for a given set of data.

Students gained some experience with regression lines in algebra 1 (see page 42). They learned the definition of a line of best fit. (It minimizes the sum of the squares of the deviations from the $y$-coordinates of the data.) They developed the conjecture that the best line always contains the centroid of the data. Also, they practiced estimating lines of best fit through trial and error.

Algebra 2 restates the definition in Chapter 1. Students begin to develop methods for calculating best-fit lines for small sets of data like this one.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>4.5</td>
</tr>
<tr>
<td>2</td>
<td>8.1</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

Algebra 2 asks students the following question.

*Of all lines of slope 3, which one best fits the data?*

Because students know

- that such a line has equation $y = 3x + b$
- the definition of best fit,

they can set up a quadratic function in $b$ that they can minimize. With this value of $b$, they can graph the best-fit line with slope 3.

They repeat this exercise for other slopes, graphing the lines each time. The lines turn out to be concurrent. Their common intersection is none other than the centroid of the data, so students strengthen their conjecture from algebra 1: Not only does the best-fit line pass through the centroid, but the best-fit line of any slope seems to contain the balance point as well.

Students assume this fact and go on to derive the best slope for the data.

In Chapter 6, students prove that the best-fit line does in fact contain the centroid.

In this way, students gain a solid grasp of what a regression line minimizes, and
they learn a method for calculating it (albeit a computationally intensive one that is intractable for large data sets). They understand what the calculator is doing when it reports a best-fit line.

In Algebra 2, the statistics work has a distinctly algebraic flavor. Consider the following problems from Algebra 2.

There are six numbers in Set $D$,

$$D = \{10, 3, 5, 10, 20, 6\}.$$  

1. Change one of the numbers in Set $D$ to make a new set with a smaller standard deviation.

2. Change one of the numbers in Set $D$ to make a new set with a larger standard deviation.

3. **Take it Further** Change one of the numbers in Set $D$ to make a new set with the same standard deviation.

To solve this problem, students experiment using the following identity, proved in Algebra 2:

$$\bar{x}^2 - \bar{x}^2 = \sigma^2$$

Algebra is again a tool for experimenting with statistics.

Students’ work in statistics can also shed new light on and add importance to some topics in algebra. For example, the following problems provide an application of the algebraic concepts of linearity and linear transformations.

1. Build Set $A$ with seven elements so that it has mean 25 and median 23.

2. Build Set $B$ so that each element in Set $B$ is seven more than each element in $A$. Find the mean and median of Set $B$.

3. Build Set $C$ so that each element in Set $C$ is twice each element in $A$. Find the mean and median of Set $C$.

4. Find two ways to build a set based on Set $A$ that has mean 1.

Students pick up on the idea that when the data set is transformed, so are the mean and median. These linear transformations of the data affect the mean and median by the same transformation.

Students later find that the same cannot be said about standard deviation. It is unaffected by translations, but scaling the data scales the standard deviation. This leads to the concept of standardizing a data set (sometimes called calculating a z-score).
What is Different About This Approach to Functions?

A huge challenge for students is seeing the connections in their mathematics work. In the CME Project, the emphasis on algebraic structure is intended to help students connect ideas.

For example, when studying arithmetic, students learn about identity elements and inverses. When they study functions, students learn about identity functions and inverses. Similarly, they can learn about identity matrices and inverses.

Students can go through an entire curriculum without a sense that these things with the same names actually have the same properties. That kind of focus is at the core of CME.

CME Project Precalculus also focuses on families of functions and the analytic features of functions and their graphs. The analytic features include continuity, limits, extreme values, and slopes of tangents. This leads to informal introductions to the derivative and integral—central concepts in calculus.

Polynomial Functions

In CME Project Precalculus, students characterize polynomial graphs and analyze their features. Students think about continuity, zeros, the signs of the function values, and the behavior of the outputs for inputs that have very large absolute value.

They also look at secant lines—lines that each intersect the graph of a polynomial function in at least two points. They explore what happens as one of the two points approaches the other and see that there is a particular tangent line at any point on the curve. (See page 70.)

Of course, the slope of this tangent to the curve at a point is the derivative, but students are not finding derivatives at this point. They are, however, developing informal understanding that will provide an excellent background for understanding what derivatives are and what it might mean to find the derivative in different contexts.

Rational, Exponential, and Logarithmic Functions

In CME Project Algebra 2, students study rational expressions—quotients of polynomials—in a purely algebraic context. They learn to simplify them. They practice adding, subtracting, multiplying, and dividing them. The rational expression

$$\frac{x^2 - 9}{x^2 - 5x + 6}$$

simplifies to

$$\frac{x + 3}{x - 2}$$

because you can cancel the common factor $x - 2$.

In Precalculus, students analyze rational functions—quotients of polynomial functions—by adapting the techniques they learned to analyze polynomial functions. The Form Implies Function Principle of Algebra 1 (page 61) does not apply to rational functions.

For formal expressions, the equation

$$\frac{x^2 - 9}{x^2 - 5x + 6} = \frac{x + 3}{x - 2}$$

holds. But the functions

$$x \mapsto \frac{x^2 - 9}{x^2 - 5x + 6}$$

and

$$x \mapsto \frac{x + 3}{x - 2}$$

are not equal because they have different domains. (The first function excludes 2 and 3 from its domain. The second function excludes only 2 from its domain.)

Students sketch graphs of rational functions and identify asymptotes and discontinuities. They also find limits as $x$ approaches $\infty$, $-\infty$, or an asymptote.
In one lesson on rational functions, students investigate a case study of rational functions of the form

\[ x \mapsto \frac{ax + b}{cx + d} \]

These “linear fractional functions” have several interesting properties, including:

- Their graphs look quite similar to each other (as long as \( c \neq 0 \)). They each have one vertical and one horizontal asymptote. Each graph has two branches. Students, in fact, make this precise by showing that every one of the graphs is “affine equivalent” to the graph of \( x \mapsto \frac{1}{x} \).
- You can pair every linear fractional function with a 2-by-2 matrix with the rule

\[ x \mapsto \frac{ax + b}{cx + d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

Algebraic structure emerges again!

Two matrices define the same function if and only if one is a scalar multiple of the other. And it turns out that the product of two matrices corresponds to the composite of the corresponding functions. Hence the algebra of linear fractional functions under composition has structural similarities to the algebra of 2-by-2 matrices under multiplication, with the extra rule that two matrices are the same if one is a scalar multiple of the other.

For exponential functions, students find equations of the tangents to the graphs for different bases. They also investigate the functional equations satisfied by exponential and logarithmic functions.

For example, if \( f(x) = 2^x \) and \( g(x) = \log_2 x \), then

- \( f(x + y) = f(x) f(y) \)
- \( g(xy) = g(x) + g(y) \)

Students investigate other functional equations in the chapter’s Project.

**Trigonometric Functions**

Again, students analyze trigonometric functions by examining the slopes of the graphs at particular points. Through experimentation with secants on the curves, they build tables of values that show estimates for the slope at each point. The relationship between cosine and sine—that the cosine function is the first derivative of the sine function—emerges as a conjecture from the table of the secant slopes of the sine function.

Students find that they can interpret each trigonometric function in two ways. In CME Project Algebra 2, cosine and sine are functions of angles. Now students see that they can also regard cosine and sine as functions of arc lengths on the unit circle.

As functions, cosine and sine take in a number (the length of an arc, defined in a particular way) and return a number. So it makes sense to say “cos 3,” where the 3 does not carry units.

This focus also helps students with some concepts they otherwise would not see until calculus. For example, why is \( \sin x \) close to \( x \) when \( x \) is small?

Students look at the graphs of cosine and sine from the perspective of analysis. Again, they look at the average rate of change between points as those points start to come together.

Very often, a textbook places unit circle and graphical representations of the cosine and sine function side by side, so that students can see some behaviors that match in both representations. Maximum points are a good example of this: if students look at the graph of sine, it has a maximum at \((\pi, 1)\). Students can see this either on the unit circle or from the graph of the function.

As a bonus, this emphasis on multiple representations allows students to do more than the traditional proofs of trigonometric identities, because the representations often suggest identities. Students conjecture and then prove (or disprove) their own trigonometric identities, using the geometry of the unit circle, the algebra of complex numbers, or the traditional calculations using basic trigonometric identities.
Again, the algebraic-structure theme emerges. Calculating with \( \cos x \) and \( \sin x \) feels like calculating with polynomials in two variables, say \( S \) and \( C \), with additional simplification rules such as \( S^2 + C^2 = 1 \). Together with the addition formulas, students generate their own trigonometric identities.

**Underlying Secant-Tangent Behavior**

For each of these families of functions,
- polynomial
- rational
- exponential
- logarithmic
- trigonometric

students look at how secant lines approach a tangent line. They develop an intuitive understanding of the slope of the graph of a function at a particular point.

Students can see this by using the TI-Nspire™ technology to model a classic thought experiment. They draw a secant line containing two points on the graph, one fixed and one moveable. They look at the limiting behavior of the secant line as the \( x \)-coordinate of the moveable point gets closer to the \( x \)-coordinate of the fixed point. With TI-Nspire, students see that the secant line does not vanish—it actually turns into a tangent line.

Students then study how this is mirrored in the algebra, starting with polynomial functions.

For example, let \( f(x) = x^3 - 2x + 1 \)

Divide \( f(x) \) by

\[
(x - 2)(x - 3) = x^2 - 5x + 6
\]

Using long division, you get

\[
f(x) = x^3 - 2x + 1 = (x - 2)(x - 3)q(x) + r(x)
\]

where \( q(x) = x + 5 \) and \( r(x) = 17x - 29 \).

Substituting \( x = 2 \) shows that \( f(2) = r(2) \).

Likewise, \( f(3) = r(3) \). In other words, \( r \) is a linear function that agrees with \( f \) at \( x = 2 \) and \( x = 3 \).

So the graph of \( y = r(x) \) must be the secant to the graph of \( y = f(x) \) between the points \((2, f(2))\) and \((3, f(3))\).

Here is the generalization, as seen in the Precalculus text.

**Theorem 3.4**

Let \( f(x) \) be a polynomial and \( a, b \in \mathbb{R} \). Write

\[
f(x) = (x - a)(x - b)q(x) + r(x)
\]

where \( r(x) \) is a linear function. Then the graph of \( y = r(x) \) is the secant to the graph of \( y = f(x) \) between \((a, f(a))\) and \((b, f(b))\).

In the above theorem, all you are really interested in is the remainder \( r(x) \). (In other words, you do not need to know what \( q(x) \) is.) And your calculator’s polyremainder function can find the remainder \( r(x) \) when it divides \( f(x) \) by \((x - a)(x - b)\).

How do you find the tangent?

Let \( f(x) = x^3 - 2x + 1 \) again. Now, suppose you want to find the tangent to its graph at \( A(2, f(2)) \). Consider a movable point \( B(b, f(b)) \) on the graph. The secant between \( A \) and \( B \) is given by the remainder when you divide \( f(x) \) by \((x - 2)(x - b)\). Now, as point \( B \) approaches point \( A \), the corresponding division by \((x - 2)(x - b)\) becomes division by \((x - 2)^2\).

The following theorem summarizes these results.

**Theorem 3.5**

Let \( f(x) \) be a polynomial and \( a \in \mathbb{R} \). Write

\[
f(x) = (x - a)^2q(x) + r(x)
\]

where \( r(x) \) is a linear function. Then the graph of \( y = r(x) \) is the tangent to the graph of \( y = f(x) \) at \((a, f(a))\).

In other words, the first two terms of this Taylor expansion give the equation of the tangent line at \( x = a \).
This same focus is brought to bear on rational functions. Students try to justify whether the methods they used to find equations for secants and tangents of polynomial functions work for rational functions, which are not continuous.

To do this, students develop an argument for the following theorem.

**Theorem 3.9**

Suppose that \( f \) is a rational function whose denominator is not zero at \( x = r \). Suppose also that you write

\[
f(x) = m + n(x - r) + p(x)(x - r)^2
\]

finding first the number \( m \) and then the number \( n \). Then the rational function \( p \) is defined at \( x = r \) and does not have any factors of \( x - r \) in its denominator.

In other words, the polynomial method does indeed suggest that the first two terms of this Taylor expansion give the equation of the tangent line at \( x = r \) as \( y = m + n(x - r) \).

**Focus on Recursion and Recursively Defined Rules**

Students often meet tables like this one.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
</tr>
</tbody>
</table>

They may describe a table like this by saying, “It starts at 3 and doubles each time.” This rule does match the table. Also, it leads to a closed-form rule. The books of the CME Project include this direction toward closed forms, but also leverage students’ natural inclinations to think recursively.

With traditional classroom technology, focusing exclusively on closed forms is the only thing that makes sense, because most handheld calculators cannot handle many steps of a recursive calculation. With newer technology that includes a function-modeling language, students can build both a recursive model and a closed-form model that correspond to the same table.

What is the payoff here?

- In some situations, the recursive rules are easier to describe and work with than closed-form rules. The recurring problem of monthly payments on a loan is a good example of this.
- Issues about the domains of functions arise naturally. Now students have two models that describe the same tables, but the two functions have quite different domains. Two questions arise from looking at two models. First, will the two models continue to agree on their shared inputs? Second, what about gaps between inputs? The gaps illustrate the fact that these two functions are not actually the same function, because they have different domains.

In CME Project *Precalculus*, students learn processes to turn recursive rules into closed-form rules. For example, students look at Fibonacci numbers as well as other one- and two-term recurrences. Students actually find a closed-form function that can generate all Fibonacci numbers. The closed form can tell a student what the 1000th Fibonacci number is. This would be hard to find, even with technology, using the usual recursive rule for generating Fibonacci numbers.

**“Breaking the Calculator” to Teach Induction**

Proof by induction often causes students problems. For example, students often complain that they are assuming what they want to prove in the induction step.

The CME Project uses the limitations of a calculator to introduce the utility of proof by induction. The TI-Nspire’s function-modeling language makes this particularly easy.
The Precalculus method of introducing induction relies on the fact that in Algebra 2, students get plenty of experience finding functions that fit tables and modeling those functions on their calculators. Students have seen many tables like this one:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

They often choose to rewrite the table this way:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1 × 2</td>
</tr>
<tr>
<td>2</td>
<td>2 × 3</td>
</tr>
<tr>
<td>3</td>
<td>3 × 4</td>
</tr>
<tr>
<td>4</td>
<td>4 × 5</td>
</tr>
</tbody>
</table>

A closed form that agrees with this table is $f(n) = n(n + 1)$. Students might also make a difference table:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

A recursively defined function that agrees with the table is

$$g(n) = \begin{cases} 0 & \text{if } n = 0 \\ g(n-1) + 2n & \text{if } n > 0 \end{cases}$$

Students can use their calculators to model both $f(n)$ and $g(n)$. They can tabulate both of them to find outputs of each function.

Questions about the domains of the function $f(n) = n(n + 1)$ and the recursively defined function $g(n)$ naturally arise, of course. But for the purpose of teaching students about mathematical induction, the big question is:

Are these two functions equal for every positive integer input?

Students can experiment with their calculators to see whether these functions agree. However, the recursively defined function places a serious burden on the calculator’s memory. At some point, the calculator seems to break, or run out of memory.

As of this writing, the TI-Nspire response is Resource Exhaustion when asked for $g(56)$. So, is $f(56) = g(56)$?

One way to check is to work as the calculator works—recursively. Find the value for

$$g(56) = g(55) + 2 \times 56$$

and then check that this equals $f(56)$.

But figuring this out can be tedious. Alternatively, students can construct the following argument.

$$g(56) = g(55) + 2 \cdot 56$$ (definition of $g$)

$$= f(55) + 2 \cdot 56$$ (“Calculator says so.”)

$$= 55 \cdot 56 + 2 \cdot 56$$ (definition of $f$)

$$= 56(55 + 2)$$ (basic rules)

$$= 56(57)$$ (arithmetic)

$$= f(56)$$ (definition of $f$)

This looks like a lot of work! But now you can ask, “What happens for $g(57)$?” The answer is below. The reasons are almost the same.

$$g(57) = g(56) + 2 \cdot 57$$

$$= f(56) + 2 \cdot 57$$ (previous step)

$$= 56 \cdot 57 + 2 \cdot 57$$

$$= 57(56 + 2)$$

$$= 57(58)$$

$$= f(57)$$

Continuing this way, students eventually get to this question.

If we know that

$$f(n - 1) = g(n - 1)$$

how can we use this to show that

$$f(n) = g(n)?$$

The argument looks just like the previous two!
\[ g(n) = g(n - 1) + 2n \]
\[ = f(n - 1) + 2n \]
\[ = (n - 1)n + 2n \]
\[ = n(n - 1 + 2) \]
\[ = n(n + 1) \]
\[ = f(n) \]

What is the use of this approach?

- Students know what they are proving. They are showing that the closed and recursive functions \( f(n) \) and \( g(n) \) match on all positive integer inputs.
- Students do not get tangled up in “assuming what we want to prove” in the induction step, because they are simply extending a tabulation by one more input.
- Students see that they cannot just check that functions are equal for every possible input. They actually need proof.

For a more detailed treatment of this approach, see “When Memory Fails.”

**Combinatorics: Counting Before Formulas**

Many consider combinatorics an art—the art of knowing *when* to do what you know *how* to do. This is because there are not many algorithms that can be applied uniformly to counting problems. And, for this reason, many students feel frustrated when tackling these problems.

In CMP *Precalculus* Chapter 4, students start with plenty of opportunities to try unfamiliar problems, to develop their own strategies, to run into blind alleys, and to make mistakes and then corrections.

The chapter immediately challenges students with three problems. It is expected that students will

- be unable to solve the problems right away
- continue to consider the problems throughout the chapter as they develop more solving strategies

**Isomorphic problems**

Early in Chapter 4, *Precalculus* introduces the approach of identifying problems that it calls *isomorphic problems*.

The words *isomorphic* and *isomorphism* have technical meanings in other branches of mathematics that are compatible with, but not exactly the same as, their uses in this chapter.

Here, *isomorphic problems* identifies problems with similar mathematical structures. What follows is a favorite example of identifying isomorphic problems taken from the end of Investigation B.

The challenge:

Explain how the following exercises are all isomorphic.

- In how many different ways can you write 20 as a sum of three positive integers?
- In how many different ways can you put 20 quarters into three jars, red, green, and blue, so that there is at least one coin in each jar?
- There are 19 people and only 2 tickets to a concert. All 19 want to go, and if someone gets a ticket, it does not matter which ticket it is. Also, one person cannot receive both tickets. How many ways are there to distribute these tickets among the 19 people?

At first glance, these exercises do not look isomorphic to students. Two of the problems seem to be about 20 objects, and one seems to be about 19. But all of these problems essentially involve choosing 2 objects from 19, and so they all can be solved using the formula \( \binom{19}{2} \).

Understanding why all three of these problems are structurally similar will help students shake the uneasy feeling they sometimes have about solving combinatorial problems.
By looking at isomorphic problems, the focus changes from the formulas to the problems, or rather the similarities among the problems. By actively searching for isomorphic problems, students can find ways to link new (and unfamiliar) problems to other problems.

_Precalculus_ also introduces simple and intuitive strategies for solving certain classes of problems. As the chapter progresses, students meet more formal notation for permutations and combinations, as well as more complicated problems.

For example, here is how _Precalculus_ introduces the Counting Principle.

**The Counting Principle**

Students develop multiple counting strategies for situations that they can count in stages. The basic principle underlying these strategies is the so-called Counting Principle, although students do not meet it by name. Rather, they apply it indirectly in two counting strategies.

In the box strategy, students translate a problem into an isomorphic problem that amounts to filling boxes with various objects, one object to each box. For example, to count how many five-digit numbers you can make from the digits 1, 2, and 4, think of five boxes. In each box, you can put a 1, 2, or 4. In how many ways can you fill the boxes?

The other counting strategy is the traditional tree approach. Both the box and the tree strategies provide students with a visualization of the stages involved in the problem.

Students also meet the solve-a-simpler-problem approach. This is more subtle, and in some ways, more abstract. It is closely aligned with recursive thinking in the sense that it requires one to

- “peel off” a stage and count the remaining stages (the simpler problem), and then
- multiply this result by the number of possibilities in the last stage

In the example of five-digit numbers, you can reason that the number of five-digit numbers using only 1, 2, and 4 is 3 times the number of four-digit numbers using only 1, 2, and 4. The thinking involves imagining the list of four-digit numbers as being already made. Then you count the number of ways (3) you can get a 5-digit number from each four-digit number.

The third investigation in Chapter 4 focuses on cementing ideas of the rest of the chapter. Students revisit the Binomial Theorem from CMP Project _Algebra 2_. It is typical for students to know that there is a connection among the entries in Pascal’s triangle, numbers of combinations, and binomial coefficients, but now they are to make the connection explicit by answering questions from multiple perspectives.

### Multiple Perspectives on Conic Sections

The theory of conic sections is a collection of ideas that tie together algebra, geometry, and the analysis of functions. Accordingly, there are several perspectives on conic sections. The goals for Chapter 6 include the following.

- Students are to see that the analytic-geometry and cone-slicing definitions for conic sections produce the same curves.
- Students are to understand that, in some sense, all the conic sections are the same.
- Interested students should see that the graph of any equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is a conic section. (The left side of this equation is called a quadratic form.)

Historically, the conic sections were regarded as slices of cones, hence their collective name. The _Precalculus_ work on conic sections begins by asking students to draw sketches of possible intersections of a plane with an infinite double cone. Visualizing the plane passing through the cone at different
tilts is one way students can see that the conic sections are the same. 

Precalculus uses a very beautiful argument, due to Germinal Dandelin, to show that the geometric (cone-slicing) definition is equivalent to the locus definitions that the students learned in geometry and algebra 2; for example, that an ellipse is the set of all points for which the sum of the distances from two fixed points is constant.

In Precalculus, students use the distance formula to transform the locus descriptions into algebraic equations. Then they manipulate the equations to produce the familiar forms of the equations of an ellipse, circle, hyperbola, and parabola.

Students also use the point-tester theme to develop the equations of conics centered at the origin (or, for the parabola, with the vertex at the origin). To do this, students solve problems like the following.

Find a if

1. $(1, a)$ is on the parabola with focus $(1, 0)$ and directrix with equation $x = -1$.
2. $(a, 1)$ is on the parabola with focus $(1, 0)$ and directrix with equation $x = -1$.

After solving these, students have to find the equation of the parabola with focus $(1, 0)$ and directrix with equation $x = -1$.

The focus of this work is twofold:

- Given some geometric information about a conic section (foci and constant sum, for example), graph it and find its equation.
- Given an equation of a conic section, find enough geometric information about it to graph it.

After working with conic sections centered at the origin (or with vertex of a parabola at the origin), students see more general conic sections.

For example:

What is the equation of the ellipse that has foci $(3, 4)$ and $(7, 4)$ and “string length” 6?

To solve this, students draw the ellipse, figure out the equation of the conic section obtained by translating this one so that its center is at the origin, and then transform their equation (as they did in Algebra 2) to get the equation of the original ellipse.

Students see one additional perspective on conic sections derived from a focus-and-directrix definition.

A parabola is the set of points that are equidistant from a point $F$ (the focus) and a line $d$ (the directrix). Let $Pd$ be the distance from point $P$ to the line $d$. Let $PF$ be the distance from point $P$ to the point $F$. The set of all points for which $\frac{PF}{Pd} = 1$ is a parabola.

But what happens if you change that ratio slightly? Say, for example, the ratio is 0.8 instead of 1. There is no reason that changing this ratio should produce a conic section, but in fact it can. Even more surprising is that this method can be used to produce all the conic sections.

This eccentricity approach provides another way to help students see that the conic sections are all related in a continuous way.

An Experiential Approach to Statistics

Several features distinguish the CME Project Precalculus approach to statistics.

- Students get plenty of practice with calculating probabilities before they see much formal development.
- The algebraic approach to some of the work connects statistics to the rest of the curriculum.
- Precalculus avoids “black boxes.” There is an algebraic justification for almost every formula introduced.

For example, Precalculus looks at the additivity of expected value and variance. Students see that for a random variable $Z$, defined as the sum $Z = X + Y$ of two independent random variables, both

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expected value and variance are additive. (Students also discover that standard deviation is not additive.)

\[
E(Z) = E(X) + E(Y) \\
\text{Var}(Z) = \text{Var}(X) + \text{Var}(Y)
\]

For a random variable \( W \) defined as the product \( W = XY \) of two independent random variables, expected value is multiplicative.

\[
E(W) = E(X) \cdot E(Y)
\]

However, variance is not.

Precalculus uses formal algebra and generating polynomials to model some situations. For example,

- The binomial \( h + t \) models the results of a coin toss.
- \( x + x^2 + x^3 + x^4 + x^5 + x^6 \) models the results of a number-cube toss.
- \( 0.2r + 0.8w \) models the results of guessing on a multiple-choice question with five possible answers. Here, 0.2 is the probability of guessing the right answer.

Students raise these polynomials to powers to calculate either the frequencies or probabilities of outcomes of repeated experiments.

Here is an example.

In an experiment, you toss three number cubes and record the sum. Because you are tossing three number cubes, you want to raise the number cubes’ polynomial to the third power. This is a simple expansion for a CAS.

\[
(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^3 = x^3 + 3x^4 + 6x^5 + 10x^6 + 15x^7 + 21x^8 + 25x^9 + 27x^{10} + 27x^{11} + 25x^{12} + 21x^{13} + 15x^{14} + 10x^{15} + 6x^{16} + 3x^{17} + x^{18}
\]

The sum of the coefficients is 216, because there are 216 equally likely outcomes in this experiment. They range from 1, 1, 1 for a sum of 3, up to 6, 6, 6 for a sum of 18.

The powers in the expansion range from 3 to 18, to reflect the possible sums.

Each coefficient represents the frequency of the corresponding outcome. The coefficient of the \( x^6 \) term is 10. That means there are 10 ways to get a sum of 6 by throwing three number cubes.

Without this polynomial power method, you would have to figure out how many ways there are to make a sum of 6 on three number cubes. Usually, you would do it by making a systematic list like this.

\[
(1, 1, 4) \quad (2, 1, 3) \quad (3, 1, 2) \quad (4, 1, 1) \\
(1, 2, 3) \quad (2, 2, 2) \quad (3, 2, 1) \\
(1, 3, 2) \quad (2, 3, 1) \\
(1, 4, 1)
\]

However, it can be difficult to be sure that you have found all the possibilities. And the task becomes even more difficult for events that include a larger number of outcomes.

You can use the expansion to calculate the probability of throwing a sum of 6 on three number cubes as \( \frac{10}{216} \).

You can also calculate more complex probabilities, such as the probability of throwing a sum that is divisible by 3.

To do this, find in the expansion the sum of all the coefficients for terms with exponents divisible by 3. Divide by the total number of outcomes.

\[
\frac{1 + 10 + 25 + 25 + 10 + 1}{216} = \frac{72}{216} = \frac{1}{3}
\]

The \( x \) in the polynomial and in the expansion is just a placeholder. It is something that is there so that you can capitalize on the properties of exponents and have a convenient way (if you have a CAS) to calculate probabilities.

A Historical Approach to the Ideas of Calculus

There are many ways to introduce students to the ideas that underlie calculus, because calculus has so many origins and employs several different ways of thinking.
CME Project Precalculus sets up a notion for differential calculus in Chapter 3 with a tangent line being the limiting figure for secant lines. This should eventually help to make sense of the mathematics of motion concepts like instantaneous velocity.

To give an experiential introduction to integral calculus, CME Project Precalculus looks at the age-old problem of calculating areas of irregular shapes.

**Limits and Continuity**

Suppose you want to find the area of the region in the first quadrant bounded by the graph of \( y = x^2 \) and the vertical lines with equations \( x = a \) and \( x = b \).

One way to do it is to approximate the area by dividing the \( x \)-axis between \( a \) and \( b \) into \( n \) pieces of equal length and erecting a rectangle over each piece with height that is the \( y \)-coordinate of the point on the curve above the left endpoint of each piece.

The sum of the areas of the rectangles approximates the area of the region.

Furthermore, as \( n \) increases, the sequence of approximations gets better and better. But each approximation is always a bit less than the area of the region.

And you can do the same thing with rectangles with heights that are the \( y \)-coordinates of the points on the curve above the right endpoint of each of the \( n \) pieces between \( a \) and \( b \). This gives a sequence of approximations—each greater than the area of the region—that also gets better and better.

Now, suppose the two sequences converge to the same limit. The area of the region—always a bit greater than the terms in one sequence and a bit less than the terms in the other—must be that common limit.

And how do you apply continuity? You have already used it implicitly when you think of a number sandwiched between an increasing sequence and a decreasing sequence that can be made as close to each other as you want. But it comes up in other ways, too.

For example, it turns out that some of the approximation techniques do not work when \( a = 0 \). You cannot use them directly to find the area from 0 to 1 under \( y = x^2 \). But you can use them to find the area from \( a \) to 1 for any small number \( a \), as close to 0 as you want.

If you assume that small changes in \( a \) produce small changes in the area (that is, the area from \( a \) to 1 is a continuous function of \( a \)), then you can use this continuity to get the area from 0 to 1.

This reasoning has, like many historical evolutions of mathematical ideas, been tipped upside down in more rigorous treatments of area. In more advanced courses, the area of the region is defined to be the common limit, if it exists. There is no \( a \) priori area.

However, CME Project Precalculus has students take the approach taken by many mathematicians over the centuries—that they are on the hunt for a number that is sandwiched between two numbers they can calculate. Students use sequences of areas of things they can find to approximate, and then find areas of more general things.

**Why Do We Use This Approach?**

Several decisions are behind the development of these ideas.

- *Precalculus* tries to keep development as historically accurate as possible, while using modern algebraic techniques.
- It is assumed (except in a very few places) that each shape investigated has an area—the student’s job is to find it.
- The arguments are not airtight proofs. Instead, they employ intuitions about limits, continuity, and area. They try to capture the spirit of the actual proofs while making use of ideas that students easily accept.
- There is no use of the formalism of limits. That level of precision is better left until students have refined and articulated the intuitive ideas.
Following Functions From Algebra 1 to Precalculus

In Algebra 1
Introduce functions as machines.
  • Domain: “inputs that the function can accept”
Use differences to find functions that match tables. Informal explorations with important ideas:
  • Constant differences ⇒ linear fit
  • Constant ratios ⇒ exponential fit
  • The hockey stick image
Build recursive models. Convert them to closed form.
Form implies function.
Three points determine a quadratic.

In Geometry
A focus on developing more formal proofs
  • Methods for conceiving proofs
  • Methods for presenting proofs
Functions defined on geometric data
  • Transforming figures using coordinates
  • Transforming figures using continuous deformations
Geometric optimization
  • Reasoning by continuity
  • “Calculus without calculus”

In Algebra 2
Determine functions from their behaviors; more precision and formality
  • Different algorithms for equal functions
  • “Domain” is context-sensitive.
  • The range need not be \( \mathbb{R} \).
Function algebra: composition and inverse
Use differences to find functions that match tables. Proofs of important results on differences
  • Constant differences ⇒ linear fit
  • Constant second differences ⇒ quadratic fit

Fundamental results about polynomials
  • Polynomial function: determined by a finite set of points
  • Counterpoint to Form Implies Function Principle, a sort of Function Implies Form Principle
  • New test for function equality
Fitting functions to tables
  • Lagrange interpolation
  • The following are equivalent for polynomial functions.
    \[ f = g \text{ on } \{1, 3, 4, 7\}. \]
    \[ f - g = 0 \text{ on } \{1, 3, 4, 7\}. \]
    \[ f - g \text{ is a (polynomial) multiple of } (x - 1)(x - 3)(x - 4)(x - 7). \]
Build recursive models; convert them to closed form.
  • Compare domains.
  • Functions equal on \( \mathbb{Z}^+ \)

In Precalculus
Different models that fit a table
  • Recursive and closed form
  • Context for mathematical induction
Extending the class of functions
  • Trigonometric functions
  • Logarithmic and exponential functions
  • Sums, products, and composites of basic functions
\( n \)th differences constant ⇒ \( n \)th-degree fit
Deeper differences

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Implementing the CME Project

Pedagogical Approaches

A Note About Classroom Organization

The CME Project does not
• prescribe classroom organization
• flag mandatory uses of technology
• make judgments about what students will find difficult

The CME Project does give information to help teachers make informed choices. It
• suggests when working in groups might be beneficial
• includes pointers—in the text—to the technology appendix
• flags exercises as optional when they are not needed for subsequent work
• describes difficulties that our students or students in field-test classes had with certain problems

Pedagogical Structures

The pedagogical structures imposed by the program materials serve the primary goal of increasing mathematical proficiency. Increasing proficiency implies a structure that
• requires students to wrestle with problems as motivation and preparation for instruction
• uses multiple contexts to support flexible knowledge transfer
• gives explicit emphasis to extracting underlying mathematical themes in these contexts to help students develop expertise in identifying “conditions of applicability”
• integrates solid skill building and knowledge acquisition with activities that require adaptive reasoning, abstraction, and problem solving

Low-Threshold, High-Ceiling Approach

One of the most important of these pedagogical structures is the low-threshold, high-ceiling approach.

Each chapter starts with activities that are accessible to all students. Each chapter ends with problems that will challenge the most advanced students. It is pleasantly surprising to see how far students often take the materials.

The low-threshold, high-ceiling design of the CME Project results in a flexibility that allows for program use in a variety of settings. In both field test and pilot, the program was used with classes that ranged in length from 45 to 90 minutes and with students at every achievement and background level.

Experience Before Formalization

In the CME Project, students grapple with mathematics before seeing formal development of the ideas.

For example, Algebra 2 opens with a collection of tables like this one.

<table>
<thead>
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<th>Input, $n$</th>
<th>Output, $A(n)$</th>
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</thead>
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<td>8</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
</tbody>
</table>

A typical algebra 2 student might look at this at first and think, “I don’t remember how to do this! I took algebra 1 ages ago!”

But we have been amazed at how much mathematics students who “don’t remember any algebra 1” can generate. It is not that all students look at the table and recognize that $f(x) = x^2 - 1$ is a function that fits. More
often, students will begin with descriptions such as “the outputs go up by 1, then 3, then 5, and then 7.” As they gain experience, they move on to more formal descriptions of the functions.

After they have done their own experimenting, they see worked-out examples.

The critical point here is the order of the presentation of the mathematics. Students do see worked-out examples, theorems, and algorithms. But they see the formal development only after they have had enough experience working with the mathematics to make sense of that formal development.

The formal development acts as a capstone for students’ learning of new ideas.

**Dialogs**

The dialogs play an important role in CME Project instruction.

Students begin each investigation grappling with unfamiliar mathematics. They then see other students—usually Tony, Sasha, and Derman—encountering the same kinds of problems, making mistakes, trying out conjectures, and talking about their ideas.

The dialogs present mathematical ideas, as well as models for approaching mathematics.

Before the field tests, we were not sure what to expect from students’ interactions with the dialogs. Indeed, early in the year, some students complained that the dialogs were foolish! But again and again, field-test teachers reported that in short order, students began to speak like the characters.

While visiting a CME Project classroom, we sat behind two students, Sara and Zoie.

*Sara and Zoie are having the following conversation.*

*Sara* I found your number. It’s −5.9.

*Sara* It’s right! Look.

*Sara goes through a process of reversing steps.*

*Sara* See?

*Sara* It’s right! Look.

*Zoie* Look.

*Zoie goes through steps on her calculator.*

*Sara* No! You can’t do that! You have to push “equals” every time.

*Zoie* No! It’s the same.

*Sara* Is it?

*Zoie* Well, it should be. But it’s not.

*Sara* Why do you think it isn’t? Because of “please my dear” stuff . . . you know . . . order of operations.

*Zoie* Oh. Yes! So it’s not the same.

*Sara looks sideways at the developer sitting behind her.*

*Sara* Eew! We sound like Tony and Sasha.

**Success for All Students**

The structure of The CME Project is built to support all students’ success in developing mathematical proficiency.

Each chapter starts with activities that are accessible to all students and ends with problems that will challenge the most advanced students. In between, it introduces abstract concepts with concrete experimentation and specific numerical examples that students extend to deeper understanding.

Field tests suggest that the materials provide all students with experiences that make them feel competent and confident, as well as experiences that are more challenging. For example, here is how one of our Algebra 1 field-test teachers summarized his experience:

The students who participated in the CME Project Algebra 1 pilot did well enough on both the Stanford 9 and state TAKS exams to earn me a very sizable bonus for the 2005–2006 school year. In
fact, I received the highest bonus in the mathematics department.

Geometry teachers often comment that my students who went through the pilot perform noticeably better than most of the students from the regular and block courses. One of my best testimonies is that several of my special education students actually performed better than ever on the state's resource test. The special education chairman pushes my name whenever she needs placements for students in algebra 1.

I learned more about algebra and mathematics than ever before by using the CME materials to do my lesson studies and presentations.

—Arnell Clayton, Houston, Texas

Practice

The CME Project takes a balanced view of the role of procedural fluency in algebraic calculations.

Simply put, technical expertise in algebraic calculations is, at the high school level, necessary but not sufficient for mathematical proficiency. Furthermore, students can develop procedural fluency in the process of producing worthwhile mathematics.

The CME Project approach to skill development is through orchestrated problem sets, an old American tradition in which students, by practicing skills such as factoring and simplifying, come to new mathematical insights. And the interplay between hand calculations and CAS environments allows you to bolster algebraic proficiency with a careful analysis of symbolic calculations.

Differentiation

As soon as you have more than one student to teach, differentiation becomes a concern in your classroom.

You have school, district, and state objectives and standards to meet in a limited amount of time. It would be much easier to do if you could count on all of your students having a certain level of background knowledge, as well as consistent work habits, communication skills, and interest in the subject.

Unfortunately, that is never what you find in the real world. Instead, you are faced with a wide variety of learners with different learning styles, levels of engagement, and widely different background knowledge and skills. The process of finding a way to engage, challenge, and advance all of your students requires a great deal of effort from you and from them, but the CME Project materials can help.

Low Threshold, High Ceiling

There are eight chapters in each book, and each chapter has a low threshold—all students can find a place to begin that is within their current ability—and a high ceiling—every student can be challenged by the situation.

Each chapter has easy entry, building on prior knowledge and carefully designed experiences. Each chapter also has several jumping-off points for more advanced work.

The texts set high expectations for all students, and expose all students to high-level questions. It is often difficult to predict which students will be the ones to do advanced work on any given day, so all students are given opportunities to think about advanced questions.

Getting Started Lessons

Students begin their study of new topics by grappling with a set of unfamiliar problems in a Getting Started lesson.

Students may choose to approach the problems in a Getting Started lesson in many different ways, depending on how they perceive the situations, and the methods they feel most comfortable working with.
Group work and class discussions of results and techniques allow all students to contribute and to get a chance to see a problem in a new way by trying out someone else's idea.

These lessons also give you a chance to learn more about your students. You can see what background knowledge they bring to the class and what their preferred strategies are. You can assess their ability to communicate their ideas and understanding.

**Teaching Lessons**

As the lessons following Getting Started formalize the mathematics of the topic, students continue to gain personal experience and exposure to other viewpoints.

They have to

- make sense of ideas for themselves
- explain their ideas verbally and in writing
- focus on their ways of thinking
- verify their own solutions
- take the initiative to critique the clarity or assumptions inherent in some of the problems in the text

To sum up, the CME Project encourages students to shift final authority away from the textbook or the teacher, and take on themselves some of the responsibility of learning.

When students speak and write about what they are learning, they create explanations at their own level of understanding. Every student can learn something from this kind of synthesis of ideas.

Interested students can also extend any of the topics in the books.

All students are continually encouraged to develop and investigate their own questions as they work. Also, there are exercises in the text that either take a detour from the core material to examine a related idea, or go beyond the expected level of understanding to stretch for a deeper result.

The four courses also include many ideas for projects that students can tackle on their own or in groups. They can take the project ideas as far as they desire.

**Poorly Performing Students**

Tests of the CME Project materials have shown that many students with a history of poor performance in mathematics can prove to be remarkably resourceful and clever when approaching problems through experimentation and analysis. These students can thrive on the CME Project's approach, perhaps precisely because it does not build strictly sequentially, depending on material they may have so far failed to learn.

**High-Performing Students**

Similarly, students who are already performing at top levels will also find challenges in the CME Project. In earlier courses, they may have succeeded by being fluent with facts, formulas, and certain styles of learning. The CME Project encourages them to develop these very same facts and formulas themselves, reasoning in ways to which they may not be accustomed in a mathematics class.

**Not More of the Same**

CME Project Algebra 1 does not begin with review and retesting of students’ abilities to compute with integers and fractions. Instead, it begins with experiments in addition and multiplication tables and with the number line that challenge students to articulate rules for computation with integers and fractions.

Students see that they are to make choices for the rules for computation with integers or with fractions, with the goal of holding on to specific properties of whole-number computation. They begin to make sense of rules that may have seemed arbitrary and hard to remember before. Students also analyze common computation algorithms until they can explain these algorithms to themselves—an activity that benefits and challenges all students.
**Build on Prior Knowledge**
Similarly, CME Project *Geometry* does not expect students to start off with a great deal of background knowledge, but challenges them to access what they already know and fill in any holes in that knowledge as part of an effort to complete new and interesting tasks. Students

- investigate experimental situations, leading them to ask questions and develop their own conjectures
- formalize language as a convenience for talking to each other clearly about what they are experiencing
- develop theorems from experience, so that those who do not remember a result from earlier study have a new situation or process that helps them make sense of the result in a deeper way

Students who may not have done well in the past have a chance to succeed. Students with excellent backgrounds have a chance to analyze, synthesize, and connect what they know.

**Assessment**
“What’s on the test?” is a student’s way of asking what is important to learn.

You should always choose tests and other assessment tools to assess what you value in learning so that your answer to that student reflects your most important goals.

Some mathematical assessment may be designed to gauge whether students have learned and can use a collection of facts, rules, and algorithms. This program advocates a deeper focus—the assessment of a student’s full mathematical power.

**Assessment Goals**
Many measures of success should be used when you assess a student’s mathematical growth. Certainly, correctness and completeness of solutions are two goals. Two others should be the quality of a student’s thinking and the growth that takes place in that thinking.

**Do the students**
- have reasons for what they did with a problem? Are their reasons sound?
- strive for and show interest in extending their knowledge and reasoning?
- explain ideas in clear and precise ways, using both the informal language of good social communication and the formal conventions of mathematics that ensure clarity and precision?
- follow and critique one another’s work? Can they develop and present a coherent mathematical argument?
- pose and pursue new problems?
- show growth in perseverance and depth of thinking?

Of course you (and they) will still care about “getting the right answer.” In mathematics, as in brain surgery and engineering, that is important. But a CME Project teacher should not value arriving at right answers above the quality of students’ thinking.

Also, the CME Project emphasizes matters of mathematical substance much more strongly than matters of convention and vocabulary, and the CME Project assessments reflect this emphasis.

**Assessment Tools**
This and subsequent sections describe several of many ways to assess student learning in the CME project and suggest how to grade.

**Problems and Discussions Embedded in the Lessons**
Lessons include many opportunities for students to try out and discuss new ideas and techniques.

- After students have seen a new technique for solving a problem in an Example, they can try it out immediately in a For You to Do exercise. Often, Minds in Action dialogs leave some key details for students to work out for themselves.
• For Discussion sections raise larger questions for students to talk about with each other. These questions may be speculative, asking students to present their initial conjectures. They may be summarizing, where the class brings together its ideas.

Problems and Exercises From the Text
The heart of the CME Project materials is what it asks the student to do. Any good problem or exercise can also function as a good assessment. We suggest that you use exercises from the text as the backbone of any assessment plan.

• Students should do Check Your Understanding exercises in class. By giving students some class time to work on these exercises, you can assess their preparation for individual work on the On Your Own exercises that follow.

• The Teacher’s Edition identifies “core exercises.” These form the heart of each lesson. Although you do not want to devote a great deal of class time to going over homework, you should find a way to evaluate students’ work frequently, and to give them opportunities to ask questions and get help.

• Use Write About It exercises for journal entries or as homework exercises that you collect for grading.

• Better students can use optional, challenging problems as a basis for a series of journal entries, where they record initial conjectures, show first attempts, revised solutions, and summaries.

• Find ways for students to see and respond to work from other students.

Journals
Journal writing reveals students’ understandings and misunderstandings, stimulates reflection, stimulates construction of knowledge, develops students’ skills in communicating mathematically, and encourages organization. If you use journals for assessment purposes, it is important that students know the criteria you will use to evaluate their journals.

A student journal might include
• responses to Write About It exercises
• a particularly interesting or important exercise, written up in detail
• a problem that initially gave the student trouble, but now is reworked and revised. The new work should include explanations of what is done differently and why.
• an individual’s explanation of a problem that was worked on in a group during class
• lists of new vocabulary words and definitions
• lists of known facts and properties
• lists of conjectures yet to be verified
• writing about affective factors—personal reactions to mathematics or to the process of learning mathematics

Portfolios
Portfolios can be effective for collecting work samples that document a student’s growth and accomplishments over time. Because portfolios contain a history of the student’s work, they are ideal for assessing the development of mathematical habits of mind.

Here are some possible steps for using a portfolio.

• At the end of a chapter, a student looks carefully back through all accumulated work—a pile of homework papers, tests, projects, unfinished problems, perhaps a journal, or printouts from the computer lab.

• From this assortment, the student selects the most important things, using criteria you provide. You can direct students to choose favorite work, important proofs, something that was revised and improved, the most difficult problem, or
simply something that shows best what the student learned.

- The student then presents the portfolio, speaking to you, other students, or parents about the contents. You may have students write a summary of their reflections on the contents and a short introduction to each piece, explaining why it was selected.

The real assessment opportunity for the student comes from making and maintaining the portfolio. Your work is to set up the initial requirements, and to evaluate the student's choices and summary. You likely will already have graded individual items in the portfolio earlier in the chapter.

**Student Presentations**

Presentations give students a chance to demonstrate a deeper understanding of a specific problem as well as more general mathematical techniques and understandings.

For presentations, students must organize their thoughts, communicate their ideas clearly, and respond to questions posed by their peers and by the teacher.

A presentation may be a report or a demonstration of a problem solution. It could even be an introduction to a topic for the class. This could include example problems, moderating class discussion, and answering questions.

Here are some suggestions for using presentations:

- Have a student submit a written version of the presentation to assure ease in grading and to encourage good preparation.
- Encourage questions. Establish an atmosphere in which anyone who is confused about a point in a presentation feels free to say so.
- Consider allowing students to assess each other by providing them with a simple, quick-to-use rubric.

**Projects**

A major goal for high school mathematics classes should be to teach students research skills in mathematics, including how to work productively on a single problem or idea for a sustained period of time. Projects are an ideal means for developing this skill.

Also, projects allow students to explore connections between mathematics and science, art, sports, or other areas of personal interest, and to show some of their other talents.

Here are some suggestions for using projects:

- You can assign individual or group projects that enrich the activities and foster exploration of mathematical topics at any time. You will find suggestions for these projects in the texts.
- Project ideas will also emerge directly from the class investigations. The text also contains historical perspectives, anecdotes, and language facts that may spark the curiosity of a student and result in a project.
- Some of the text projects involve working through a proof of a major result, understanding each step in the process, and making sense of the argument as a whole.
- You can allow students to make a project of any proof that will stretch their abilities. They can write up the details they have gleaned from class work or individual study of an argument in the text.
- You can also use a project effectively as a culminating assessment activity for an investigation or chapter. In the project, the student demonstrates mastery of the major mathematical ideas in the section.
- You may find it helpful to consult with individual students about their projects periodically, to assign intermediate deadlines for accomplishing various parts of the project, and to give students a written explanation of the assessment guidelines.
Implementing the CME Project

Use of Technology

Students in the CME Project use technology in many of the same ways that other students use technology.

- To test conjectures
- To reduce computational drudgery
- To graph equations and functions
- To perform statistical analyses on data
- To provide examples of theorems and results

CME Project students make other uses of technology that are less standard. They use the calculator as a context—figuring out what is behind the calculator’s built-in functions. For example, one lesson helps students understand the mathematics that underlies the calculator functions that compute standard deviation, variance, mean, and best-fit lines.

To be more specific, there are three overarching uses of technology in the CME Project.

- Computers and calculators make tractable and enhance many beautiful classical topics, historically considered too technical for high school students.
- Technology also supports experiments with mathematical objects—numbers, algebraic expressions, geometric figures, and mathematical functions.
- Technology builds computational models of mathematical structures that have no faithful physical counterparts.

The CME Project’s choice of specific technologies is consistent with the Project’s focus on habits of mind.

- To support reasoning by continuity, the CME Project uses graphing technology and dynamic geometry software.
- To help develop functional and algorithmic thinking, the CME Project uses a function-modeling language (described below) and a spreadsheet.
- To help develop a sense for mathematical structures and the habit of looking for structural similarities in algebra 2 and precalculus, the CME Project uses a computer algebra system (CAS).

Many schools and teachers have access to these environments as separate packages (graphing calculators, spreadsheets, and so on). However, Texas Instruments’ TI-Nspire™ technology bundles together all of these environments. What’s more, these separate environments “talk” to each other in the TI-Nspire (data generated in a geometry experiment can be gathered in a spreadsheet, for example), thus making this handheld device ideal for the purposes of the CME Project.

Accordingly, each Student Edition has a TI-Nspire Technology Handbook in the back. It is there to help students work with the TI-Nspire device on explicit examples called out from the text.

Example 1: Building Models for Functions

New high-end mathematical calculators—including the TI-Nspire and most computer algebra systems—contain a capability that the CME Project calls a function-modeling language (FML). What this means is that students can make user-defined functions—here called models—in a language that is quite close to ordinary mathematical notation. Then they can use the models as if they are built-in functions.

For example, to build a model of the function $f$ defined by

$$f(x) = x^2 + 1,$$

you can enter

define $f(x) = x^2 + 1$
Once you define this model, you can use it as if it were a primitive:

- You can evaluate \( f \) at inputs.

```
1.1 RAD AUTO REAL
Define \( f(x) = x^2 + 1 \)
```

- You can graph \( f \). On the TI-Nspire, you can type \( y = f(x) \) in a text box and drag it to one of the axes.

```
1.1 RAD AUTO REAL
Define \( f(x) = x^2 + 1 \)
```

You get the graph of \( y = f(x) \).

- You can tabulate \( f \) in a spreadsheet and manipulate the data.

```
1.1 RAD AUTO REAL
Define \( f(x) = x^2 + 1 \)
```

The TI-Nspire tabulates function values and generates first differences.

- You can compose \( f \) with other functions, built in or user-defined.

```
1.1 RAD AUTO REAL
Define \( f(x) = x^2 + 1 \)
f\{2+3\} 785
\sqrt{2} 3
\{\text{int}\{x\}\} 3
\{2\cdot \phi + 1\} 4 \cdot \phi^2 + 4 \cdot \phi + 2
```

So, you can calculate \( f(\sqrt{2}) \). Or, if your computer supports computer algebra, you can even ask for \( f(2\phi + 1) \). The system will produce some version (simplified or not) of \( (2\phi + 1)^2 + 1 \).

In *Algebra 2*, students study a table like this. Each \( \Delta \) column is the column of successive differences of the column to its left.

<table>
<thead>
<tr>
<th>Input, ( x )</th>
<th>Output, ( f(x) )</th>
<th>( \Delta )</th>
<th>( \Delta^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>30</td>
<td>14</td>
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<tr>
<td>3</td>
<td>46</td>
<td>44</td>
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<tr>
<td>4</td>
<td>90</td>
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</tbody>
</table>

Students find that they can match the outputs with a quadratic function of the following form.

\[ f(x) = 7x^2 + \text{ terms of lower degree} \]

At this point, students can build a model for \( g(x) = 7x^2 \), tabulate it next to the outputs of \( f \), and subtract the outputs of \( g \) from the outputs in the table. Then they can use the methods of the text to figure out the rest of a formula for \( f(x) \).

But students can also do this easily by hand, so what value is added?
By building the model in a language that is faithful to mathematical notation, students build an image in their minds of the function $g$. When students build computational models of mathematical functions, they are reviewing, expressing, and getting a chance to examine their own ideas about these functions.

At one level, they are getting the benefit that generally comes from writing out one’s own ideas carefully and in detail. That process, by itself, helps organize one’s thinking.

Without computational technology, students have to be satisfied with their written notes. The students who can bring these notes to life entirely in their heads have more success than those for whom the notes just sit motionless on the paper.

But when the notes are executable on a calculator, students can run the models they have made, verify their correctness or debug them, and even use them as parts of more complex models.

Students who are not yet skilled enough to hold many parts of a model in their heads can build the parts one by one, show how they go together and, for the present, leave the orchestration to the calculator or computer. In short, computers can help students tinker with the mechanics of mathematics.

Most FMLs are rich enough to support piecewise-defined and recursively defined functions. The recursively defined models are ideal for some of what occurs in the program. For example, you can define the function $f(x) = 3x + 7$ recursively (on nonnegative integers) as

$$f(x) = \begin{cases} 7, & \text{if } x = 0 \\ f(x-1) + 3, & \text{if } x > 0 \end{cases}$$

CME Project Algebra 1 introduces this notation. Algebra 2 uses it extensively.

TI-Nspire has templates that look almost exactly the same.

Building this model, analyzing it, and experimenting with it helps students with certain key ideas.

- It helps students think of functions as things in their own right.
- It helps students gain facility with mathematical case notation.
- Students learn to compare the closed form to the recursive form. This leads to questions about domain.
- In a given situation, it is often easier to see a recursive pattern than a closed-form pattern (monthly payments on a loan, for example). Students can then tabulate the recursively defined model and look for closed-form models.

**Example 2: Reasoning by Continuity**

Much of the thinking involved in calculus, physics, and analysis involves reasoning about systems that change in a continuous way. The thought experiments involved in this kind of work include imagining change, looking at extreme cases, and seeking invariants in changing systems.

Dynamic geometry (DG) environments can help students develop these analytic habits of mind. DG environments support computer experiments that mirror the thought experiments of mathematicians and scientists.

For example, a classical theorem in geometry states that if two chords intersect inside
a circle, the product of the lengths of the segments of one chord is the same as product of the lengths of the segments of the other.

\[ AP \cdot BP = CP \cdot DP \]

Instead of thinking of two chords, think of a variable chord \( AB \), fixed at \( P \), so that you can drag \( A \) around the circle. As you move \( A \) around the circle, \( AP \) and \( BP \) change, but their product remains invariant.

\[ AP \cdot BP \text{ is invariant.} \]

The fact that the product \( AP \cdot BP \) is invariant as you move \( A \) around the circle means that this invariant product is a function of \( P \). It does not depend on the particular chord (the position of \( A \)) used to compute it.

Because this shift of emphasis allows you to define the invariant as a function of \( P \), you are now open to investigating the properties of this function and how it extends to other domains (for example to points on or outside the circle).

Another example comes from an optimization problem: Where should you place a point inside a triangle so that the sum of the distances from the point to the three vertices is as small as possible?

Minimize \( AD + BD + CD \).

DG environments allow students to “clone” the lengths \( AD, BD, \) and \( CD \) end to end on a segment and to measure the resulting total.

\[ 23.97 \text{ cm} \]

Students now have a laboratory in which they can experiment with the function

\[ D \rightarrow AD + BD + CD \]

and they can make conjectures about the optimal position for \( D \).

**Example 3: Experiments in Algebra**

Experienced users of mathematics see the objects of mathematics, functions, algebraic expressions, and geometric relationships as real phenomena—objects with which one can experiment and conjecture. Beginning students need help developing these habits. For algebraic phenomena, a computer algebra system (CAS) is an ideal tool to “make real” polynomials and formal expressions.
For example, the factor game is a popular activity in middle school mathematics. You give students a table of the integers between 1 and 30.

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<thead>
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<th></th>
<th>1</th>
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<th>5</th>
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<td></td>
</tr>
</tbody>
</table>

Two people play the game. Player A picks a number from the table, gets that many points, and crosses off the number. Player B gets the sum of all the factors of Player A’s number that have not yet been crossed off. (If all the factors of Player A’s number have been crossed off, Player A gets no points.)

Players then switch roles and Player B picks a number.

The CME Project version of the factor game has the same rules, but it involves polynomials.

Example 4: Investigating Algebraic Structure
Polynomials are universal objects in algebra. You can model many algebraic systems, such as the complex numbers, with polynomial arithmetic having some extra simplification rules. You can use a CAS to build such models.

In CME Project Precalculus, students study roots of unity—roots of the equations \( x^n - 1 = 0 \) for positive integers \( n \). Using De Moivre’s theorem, they derive explicit formulas for these roots that involve trigonometric functions. But they also study how the TI-Nspire’s CAS can model the arithmetic of these complex numbers.

For example, in one investigation, students are studying the roots of \( x^7 - 1 = 0 \), one of which is the number \( \zeta \) defined by

\[
\zeta = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}
\]

In the process of finding something else, students must find the value of a complicated-looking expression:

\[
(\zeta + \zeta^6)(\zeta^2 + \zeta^5) + (\zeta + \zeta^6)(\zeta^3 + \zeta^4) + (\zeta^2 + \zeta^5)(\zeta^3 + \zeta^4)
\]

What follows is from Precalculus, Lesson 2.12.

\[
\ldots \ldots \text{didn't it feel as if you were just expanding this polynomial}
\]

\[
(x + x^6)(x^2 + x^5) + (x + x^6)(x^3 + x^4) + (x^2 + x^5)(x^3 + x^4)
\]

\[
\text{to find its normal form?}
\]
You could have just entered the above expression in your CAS, getting
\[ x^3 + x^4 + 2x^5 + 2x^6 + 2x^8 + 2x^9 + x^{10} + x^{11} \]
and replacing \( x \) with \( \zeta \) in the result. Then you replaced high powers of \( \zeta \) (powers 8, 9, 10, and 11) with lower ones, using the relation \( \zeta^7 = 1 \).

But there is an even better idea here. Suppose you take the above polynomial and divide it by \( x^7 - 1 \). You get
\[ x^3 + x^4 + 2x^5 + 2x^6 + 2x^8 + 2x^9 + x^{10} + x^{11} = (x^7 - 1)(2x + 2x^2 + x^3 + x^4) + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 \]
Now replace \( x \) with \( \zeta \) on both sides.
Since \( \zeta^7 - 1 = 0 \), you get
\[ \zeta^3 + \zeta^4 + 2\zeta^5 + 2\zeta^6 + 2\zeta^8 + 2\zeta^9 + \zeta^{10} + \zeta^{11} = (\zeta^7 - 1)(2\zeta + 2\zeta^2 + \zeta^3 + \zeta^4) + 2\zeta + 2\zeta^2 + 2\zeta^3 + 2\zeta^4 + 2\zeta^5 + 2\zeta^6 = 2\zeta + 2\zeta^2 + 2\zeta^3 + 2\zeta^4 + 2\zeta^5 + 2\zeta^6 \]

If you think about it, this process amounts to
- dividing the expression
  \[ x^3 + x^4 + 2x^5 + 2x^6 + 2x^8 + 2x^9 + x^{10} + x^{11} \]
  by \( x^7 - 1 \)
- taking the remainder
- replacing \( x \) with \( \zeta \) in the remainder
- simplifying the result

Most CAS systems allow you to get the remainder directly with a built-in function.

This idea of modeling algebraic systems (like the system of the 7th roots of unity) with remainder arithmetic for polynomials is central to many mathematical disciplines and to the design of computer algebra systems.

The use of a CAS allows high school students to get a glimpse of the power of this method and to experiment with many different types of algebraic systems.
Implementing the CME Project
Teacher Materials

Each course comes with a variety of instructional materials and tools.

Instructional Materials and Tools
The CME Project components for each course consist of

Student Edition
This hardbound text contains eight chapters. Each chapter contains 3–4 Investigations. Each Investigation contains 3–6 Lessons. Each chapter also includes a Mid-Chapter Test, Project, Chapter Review, and Chapter Test. Even-numbered chapters end with a Cumulative Review.

The end-of-book materials include a TI-Nspire™ Technology Handbook to support students’ use of this new technology.

Teacher’s Edition
The structure of the CME Project Teacher’s Edition provides the support needed at the point of use.

• Detailed mathematics background for every chapter, investigation, and lesson
• Complete support for pacing and resource management
• Detailed, daily teaching plans
• Answers for problems and exercises found within the wrap on the same page as the student pages
• Support for differentiating instruction
• Assignment Guide

Practice Workbook
This booklet provides additional practice for every teaching lesson.

Assessment Resources
All program assessments, including lesson quizzes and mid-chapter, chapter, quarter, and mid-year tests, are in this one booklet.

Teaching Resources
This booklet includes blackline masters of resources that can be used as transparencies or handouts, as well as answers to the Practice Workbook exercises.

Solutions Manual
This booklet contains worked-out solutions for every exercise in the following sections of the Student Edition.

• For You to Explore
• Check Your Understanding
• On Your Own
• Maintain Your Skills
• Mathematical Reflections
• Mid-Chapter Test
• Chapter Review
• Chapter Test
• Cumulative Review

TeacherEXPRESS CD-ROM
This resource (one per grade level) contains a lesson planner and access to an electronic version of all program resources. This tool also contains correlations to state standards so that users can keep a running record of concepts covered in class in preparation for high-stakes tests.

Online Resources
• Mathematical Background: This information is available online to educate parents about the CME Project. It includes an overview to help parents
understand the structure and goals of the CME Project and learn how the mathematics develops throughout the program.

- **Data Update**: Some topics include links to a Web site that can provide the students with current data on that topic.
- **Link to TI Resources**: Students and teachers have access to online tutorials developed by Texas Instruments to support the use of TI-Nspire technology in the classroom.

**Teacher’s Edition**

The Teacher’s Edition design considered the input of several groups of teachers in focus groups. It consists of a student text wrapped within bottom and side-column commentary, annotations, and answers for teachers. To summarize what is in the wrap:

For each chapter, there is a Chapter Opener, which consists of

- **Chapter Overview**
- **Mathematical Background**: This explains any novel approaches in this curriculum or describes the development during the course for some of the topics in the chapter. It may also have some more advanced mathematics for teachers.
- **Developing Students’ Habits of Mind**: This lists the key habits of mind for the chapter and describes how they figure in the chapter.
- **Road Map**: This is an investigation-by-investigation description of the chapter.
- **Recommended Pacing**: This gives the number of days for each lesson, materials, core and optional exercises for in-class work and for homework, and any special notes.

Each Investigation has an Investigation Opener consisting of

- **Overview of the Investigation**
- **Learning Goals**
- **Habits and Skills**
- **Road Map**: This is a lesson-by-lesson description of the investigation.

For each Lesson, there is

- **Lesson Overview**
- **A side note listing materials, vocabulary, and the core and optional exercises for in-class work and homework**
- A sample lesson plan including Launch, During the Lesson, and Wrap Up suggestions
- Notes on suggested discussions and lesson activities
- Answers to any questions asked during the lesson (For You to Do, In-Class Experiment, etc.)
- Answers to all the exercises

**Assessment**

The CME Project provides a variety of assessment options and support.

- As part of the Lesson Plan for each lesson of the CME Project, the Teacher’s Edition includes lists of **Learning Goals and Habits and Skills**. These help the teacher focus the lesson on the most important content. This focus is especially important in a curriculum that emphasizes experience before formalization, because some questions and exercises in a lesson may preview formal work to come later. Other questions will assess student mastery of a particular concept.
- Each lesson also includes **For You to Explore or Check Your Understanding** exercises. The teacher should assign these to be completed in class. They give the teacher a measure of student understanding of the day’s lesson and a first chance for intervention.
- Other formative assessment items in the lesson exposition include **For You to Do** problems, **For Discussion** questions, and
In-Class Experiments, each with answers in the Teacher's Edition. These provide checkpoints for the teacher to evaluate student understanding during the course of a lesson.

- **Lesson Quizzes** with answers are in the Assessment Resources (with facsimiles in the Teacher’s Edition) for a more formal assessment at the lesson level.
- To help students assess their own progress, each chapter in the Student Edition includes a sample *Mid-Chapter Test* and a sample Chapter Test. Even-numbered chapters conclude with a *Cumulative Review*.
- Each chapter also includes a **Project**. The Projects allow teachers to assess a student’s ability to investigate a mathematical situation in greater depth and to present a mathematical argument in writing.
- The Assessment Resources booklet for teachers contains two forms of the *Mid-Chapter Tests* and *Chapter Tests*. The Assessment Resources also includes *Quarter Tests* for each of the four quarters and two forms of cumulative *Mid-Year Tests* and *End-of-Year Tests*. Answers for all assessment instruments are included in their respective publications.
- Each investigation within a chapter of the Student Edition begins with three framing questions, *Learning Goals*, and *Habits and Skills* for the investigation as a whole, to help students focus their studies.
- Each investigation closes with *Mathematical Reflections*, an exercise set that repeats the Investigation-opening framing questions. This gives students an opportunity to put ideas together for themselves after they see formal development of the mathematics. The Teacher’s Edition includes answers to the Mathematical Reflections.
- **ExamView** is software that comes with the CME Project and allows teachers to make leveled practice worksheets quickly and easily for targeted intervention.
A Teacher’s Perspective on the CME Project

Hi again. I’m Annette Roskam and I’ve told you part of my story—the most exciting part—on page 2 of this guide. Let me tell you more.

I had the distinct privilege of teaching CME Project Algebra 1 for two years during its development. My algebra 1 classroom in rural, medium-sized Rice Lake High School in northwest Wisconsin had students from grades 9–11 whose skill, abilities, and interests in mathematics varied widely.

I want to share with you some of my struggles and solutions, and some of the amazing experiences my students and I had. Since I faced the most challenges in adjusting to this new curriculum, I will focus on the first half of the course.

I was able to move more easily through the second half of the course, once my students and I had established an environment in which we could develop our mathematical habits of mind.

Before I began my first year with the CME Project, I attended a professional development seminar led by the authors. Then, throughout the year, I tried to keep the ideas from the seminar in mind.

• Don’t get bogged down in ultra-detailed reasons for problems, because the formal algebraic ideas will come later.
• Allow and help students to begin articulating their ideas using their own words.
• Keep big ideas in mind, and make sure students understand why the arithmetic rules they learned in elementary and middle school work.
• Help students learn to articulate their thinking clearly and to think deeply about mathematical ideas without being told every detail up front.

Once we moved on to solving first-degree equations, many students intuitively began thinking about transforming equations using the properties of equality. They did this without formal teaching as they figured out Spiro the Spectacular’s number tricks. By the time Lesson 2.11 introduced the formal properties, my students were ready.

My students successfully solved increasingly complicated equations earlier than my previous classes had with other approaches. The students who returned to my classroom for algebra 2 the next year correctly suggested how they might apply these mathematical ideas to new situations involving quadratic equations.

Teaching my students to write equations from word problems has always posed a special challenge for me as a teacher. It seemed that the strategies I had tried—making a list of words that indicate particular operations, using a chart to “fill in the blank” for trite problems that all had the same setup—never carried my students through into subsequent courses.

I still had to be patient and step back—which was, at times, so hard—to let my students grapple with finding equations that modeled word problems. The CME Project’s guess-check-generalize approach helped my students to

• get comfortable trying something
• modify their guess until it worked
organize their thinking
• see the purpose for creating an efficient rule (equation) to model a situation

I saw my students rise to the challenge of thinking about the mathematics rather than trying to memorize some set of steps for solving the problem.

One of the challenges I faced in teaching this chapter was an ingrained sense that my students needed to learn to use the properties of equality to solve equations. Coming from a very procedural background, I had a hard time letting go of this and letting my students play or, as Dr. Cuoco and his team would say, “muck around in the mathematics.”

I wanted to tell my students how to solve the problems (and they wanted this too), but the more I stepped back and asked questions instead of giving answers, the more I saw my students begin, albeit gradually, to ask their own questions and try out their own ideas. This was a challenging, and at times painful, transition for all of us, especially coming from a climate of “math teacher tells, students follow.”

Although some of my students (and their parents) remained uncomfortable with the question-asking stance I took, I truly believe the rewards outweighed the initial agony. I repeatedly saw my students, both during this course and in subsequent mathematics courses, improve their ability to anticipate how mathematical principles might be applied to new situations. It gave me confidence and satisfaction to see my students correctly apply what they had already learned about algebra to new situations.

This was definitely exciting!

I’ve already shared the best moment on page 2, but let me also share the second-coolest experience in my entire teaching career.

One day when my students were working on a lesson on solving equations, one student, Cody, asked me whether his solution was correct. It was early in the curriculum, my first year. I was nervous because the materials were challenging for my students, and I was still getting my head wrapped around the idea of letting my students muck around in the mathematics rather than directly teaching the procedures for solving equations.

I tried to phrase a response in an encouraging manner, but one that left Cody in charge of correcting his work. I said, “Well, you have one thing to change in your work to have the correct answer.”

I waited for him to retort, “Why don't you just tell me what I need to fix?”

Instead, to my delight, he responded, “Wait! Don't tell me. I want to figure it out myself.”

Shortly, he returned with his work fixed, proudly stating what he had changed and why his work was now correct.

We were both elated, to say the least.

Not too long after this interaction in class, Cody’s dad came to our parent conferences. He asked me how Cody was doing in class and what he was learning. He seemed skeptical, especially since he knew only my class was using this new curriculum.
I explained to him that this curriculum would teach his son the same content as the district-adopted curriculum, but it would also teach him more—about how to think like a mathematician, articulate his ideas, and be able to predict how new mathematical ideas he encountered might work. I said the curriculum was designed to let students grapple with the mathematics before bringing ideas to closure, but that we would always get there.

The father nodded and said, “Okay. We'll see how it goes.”

I next saw Cody’s father the following fall. He firmly shook my hand and said that he wanted to thank me because I had taught his son how to think, and no one had done that before.

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By this time, I had also begun responding to my students’ questions and ideas with statements such as,

“Hmm, I don’t know. (Of course, I knew!) Let’s try it and see,”

“That’s interesting, tell me more,” and

“Why do you think that will work?”

Although they did not like this approach for quite a while, they learned that they needed to justify their ideas and that I would rarely give them a quick answer.

When we worked on graphing, my students grappled with the difference between the results in a table (and on the graph) of $x \rightarrow x + 2$ and $y \rightarrow y - 5$, moving the graph right 2 and down 5, compared with what appears to be similar moves in an equation, for example, $y = 3x - 1$ and $y - 5 = 3(x + 2) - 1$.

I looked for opportunities to encourage them to think about possible distinctions between these forms (table or graph and equation) and reasons for them.

It was one of my lower-level students who suggested rewriting equations in $y = \text{form}$ in order to more easily build our table of values to graph lines. The equations as point-testers tied her idea back to the relationship between the table, equation, and graph and helped to build a basis for writing equations of lines in the next chapter.

By the time my students encountered lines in CME Project Algebra 1, they had made sense of how the transformation indicated in the equation showed up in the graph, and vice versa.

The presentations of transformations in Chapter 3 proved to be a powerful tool for making this important idea explicit for my students.

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Sasha and Tony’s dialogs helped break the ice for talking mathematics for some of my students. Reading the dialogs quietly to themselves, in pairs, or even acting them out for the whole class, and also writing responses to the key questions for each investigation, allowed my students to continually model mathematical discourse and to explain their thinking verbally and in writing.

One day two girls excitedly came to class with a story of how they had had a “Tony and Sasha conversation over the weekend.” They knew they had
experienced firsthand how these two CME Project characters worked their way to a plausible solution to a genuine mathematical problem they had faced.

I saw my students take charge of their own learning in Chapter 4 with the bellweather moment described on page 2. In this chapter, my students followed a natural progression of thinking, and they correctly constructed how to graph lines more efficiently than by making a table of values and plotting points.

By paying careful attention to what I said, I had become accustomed to asking questions that encouraged my students to try out their ideas and defend them with mathematical principles. Their ideas gelled. In two days, they accomplished what normally took two weeks. They were writing equations of lines using three different approaches, point-tester, slope-intercept, and point-slope forms.

This most difficult of all algebra 1 lessons, writing equations of lines, became a matter of helping my students make connections among ideas with which they were already becoming comfortable. I did not have to tell them the forms of the equations, how to decide which to use based on the information they were given, how and where to substitute which information, or how to simplify.

This group of students demonstrated their growing understanding by how they talked about using the equation of the line and the information they had. They showed me they understood that different forms of equations reveal different yet important information about the line, and that each form still produces the same line as the other forms.

I got to step to the side! I was able to do this more and more frequently during the rest of the school year, letting students take control of the learning, and guiding occasionally by asking them to describe what they were thinking.

My patience began paying off. It took half the year, but from there on the students in all of my field-test courses began to offer ideas that we evaluated, interpreted, and applied as a class. More often than not, the students were on the right track.

Reducing my direct answers and increasing my asking students to explain how or why the idea fit with other mathematical ideas they had learned, helped my students improve their ability to articulate their thinking, determine whether their ideas seemed reasonable, and take steps to align their thinking with mathematical principles they already knew.

Simplifying exponential expressions and working with radicals was nearly as difficult for my students to learn as writing equations of lines. By the time we reached the second half of the year, however, my students were used to mucking around before they determined the mathematical principles inherent in the work they were doing.

Approaching exponents and radicals the same way as I had other topics throughout the year, by

• stepping aside
• asking my students to explain their thinking
• asking them to supply new, similar problems to determine whether an idea still held water
• modifying the idea
• trying new problems
• settling on how mathematics worked

enabled my students to move fairly comfortably through the rest of the year.

This course may be the first time some students have had to think about mathematics and how it might work without first being told all the steps and rules. I tried to be patient and clear with my students (and their parents). I expected them to think first, and we would all work together to figure out the mathematical principles at work in the lessons.

Using this approach to algebra, my students appeared to understand mathematics better than if they were just told the steps to solve the problems. I rejoiced with progress.

Well-meaning parents may be concerned that their students are “fooling around” with mathematics rather than learning mathematical principles outright, and they may instruct their children in their own ways of solving equations. I reminded parents of the purpose of this curriculum: to develop mathematical habits of mind.

I also shared with them research that has shown that when students grapple with purposefully and well-constructed problems, their interest and readiness to learn increases. Further, the formal algebraic ideas they had learned in school were still embedded in this curriculum, and their children were getting the procedures they remembered from algebra along with so much more.

Although transitioning to a classroom environment of inquiry from a classroom of “teacher telling” posed challenges at times, the dividends appeared in my students’ improved abilities to articulate their thinking, engage in mathematical discourse, and anticipate correctly how they might use the mathematics they already knew in novel situations. My students became more comfortable attempting new problems and modifying their approaches rather than just giving up.

I am so glad I did not just give up, either, on different pedagogical approaches when I found them new.

I encourage you to persist, as well, as you embark on your endeavor to use the CME Project curriculum. It is never easy to encounter a new curriculum or implement new approaches at first. Let the dream of the payoff sustain you: watching students truly engage in mathematics and really develop mathematical habits of mind.

—Annette Roskam

Editor’s Note: Annette Roskam field-tested the CME Project Algebra 1 materials and revised Algebra 1 materials over two years at Rice Lake High School. She is working toward her doctorate in Mathematics Education at the University of Delaware. In the long term, she expects to work with teachers on ways to improve their efforts at improving their students’ learning.
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